Notes on Étale Fundamental Group

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Time: Fall 2015.

Étale Fundamental Group

References ① Lenstra, Galois Thy.
② SGA 1.
③ Murre, Lectures ... fundamental gp.

$F : \mathcal{C} \to \text{Sets}$. $\quad \text{Aut}(F) \overset{i}{\hookrightarrow} \prod_{x \in Ob(\mathcal{C})} \text{Aut}(F(x))$

$F(X)$ always finite

Claim: Image closed. If $(t_x)_{x \in \mathcal{C}} \notin Im(i)$, then $\exists f : X \to X'$ in $\mathcal{C}$ and $x \in F(X)$, s.t. $F(f)(t_X(x)) \neq t_{X'} F(f)(x)$.

So $\text{Aut}(F)$ has a natural structure of profinite gp.

Galois Cats

~~Galois~~ Top. gps $\rightleftarrows$ Cats $\mathcal{C}$ w/ $F : \mathcal{C} \to \text{Sets}$.

Notation: G top. gps. G-set denote the cat. Obj are (X, a). X discrete, $a : G \times X \to X$ cont.

$\hat{G} = \lim_{\substack{U \trianglelefteq G \\ \text{open}, [G:U] < \infty}} G/U$. univ. property $G \to H$ cont. H profinite then $\exists!$ $G \to \hat{G} \to H$.

Prop. F: finite G-sets $\to$ Sets. Then $\hat{G} = \text{Aut}(F)$.

pf.
- $\exists\ G \to \text{Aut}(F)$ by left multiplication
- this is cont.
- get $\hat{G} \overset{i}{\hookrightarrow} \text{Aut}(F)$ by univ. property and

$$\begin{tikzcd} \hat{G} \arrow[r] \arrow[d] & \text{Aut}(F) \arrow[d] \\ G/U \arrow[r, hook] & \text{Aut}(F(G/U)) \end{tikzcd}$$

- w/ a bit of work to show $im(i)$ is dense, as G/U is a cofinal system in finite G sets

Lemma Any exact functor F: finite G-sets $\longrightarrow$ Sets w/ $F(X)$ finite $\forall X$, is isomorphic to the forgetful functor.
pf. leave as an exercise.

Def. exact means, let $F: \mathcal{A} \to \mathcal{B}$.
① If $\mathcal{A}$ has finite limits ($\Leftrightarrow \mathcal{A}$ has a final obj $*_A$ and fibre product)
then F commutes ······ $\Leftrightarrow$, F is left exact.
② If $\mathcal{A}$ has finite colim ($\Leftrightarrow \mathcal{A}$ has a initial ϕ_A and pushouts)
... F ······ $\Leftrightarrow F$ is right exact.
③ F is exact $\Leftrightarrow$ left & right exact

Defn X connected $\Leftrightarrow X$ is not initial and any subobj. ($Y \to X$ monom.) is either isom. to $\phi \longrightarrow X$ or $X \xrightarrow{id.} X$.

Defn $F: \mathcal{C}$ - Sets functor, $(\mathcal{C}, F)$ is a Galois Cat. if
① $\mathcal{C}$ has finite lim and colim
② every obj. of $\mathcal{C}$ is a finite coprod. of connected obj
③ F has finite values
④ F is exact and [$F(f)$ isom $\Leftrightarrow f$ isom].

e.g & fact $a, b: X \longrightarrow Y$ in $\mathcal{C}$, X connected, $F(a)(x) = F(b)(x)$, for some $x \in F(X) \Rightarrow a = b$.
pf. F comm. w/ equalizer $\Rightarrow x \in Eq(F(a), F(b)) = F(Eq(a, b))$
$\Rightarrow Eq(a,b) \neq \phi$, hence $Eq(a,b) = X$.

Rmk on this Lemma, moreover, we fix $y \in F(Y)$, we see that
$\forall x \in F(X)$, $\exists f: Y \to X$, s.t. $F(f)(y) = (x)$.

Cor. $\# Aut_{\mathcal{C}}(X) \leq \# F(X)$ for X connected.

Defn. We say $X \in Ob(\mathcal{C})$ is Galois if X is conn. and '$=$'.

Lemma. For any connected X, there $\exists$ Galois obj. Y and $Y \xrightarrow{f} X$.
pf. $F(X) = \{x_1, \cdots, x_n\}$. Write $S_n \circlearrowright X^n = \coprod_{t \in T} Z_t$
pick $t \in T$, s.t. $\xi = (x_1, \cdots, x_n) \in F(Z_t)$.
~~Win if we can prove $F(Z_t) = \{(x_1', \cdots, x_n') \in F(X)^n \mid x_i'$ distinct$\}$~~
~~$\forall (x_{\sigma(i)})$, $\sigma(Z_t) = Z_t$, and $F(\sigma)(\xi) = (x_{\sigma(i)})$~~
~~So it remains to show $\forall \eta \in F(Z_t)$, coords of η distinct~~
~~But it follows from If $\eta_i = \eta_j$, then consider~~
~~$pr_i, pr_j : Z_t \rightrightarrows X$, $x = \xi \in F(Z_t)$.~~
~~By the fact before, we win! (Moreover, we see $F(f)$ is surj.)~~

only have → to prove all elts in $F(Z_t)$ have different coord.

Lemma. $(\mathcal{C}, F)$ Galois cat. Action of $Aut(F)$ on $F(X)$ is transitive for all $X \in Ob(\mathcal{C})$ connected.
pf. I = Set of isom. classes of Galois objects.
$i \in I \rightsquigarrow X_i$ representative, choose for each i, $x_i \in F(X_i)$
$i \geq i' \Leftrightarrow \exists$ morphism $X_i \longrightarrow X_{i'}$. and in this case
$\exists! \ f_{ii'}: X_i \longrightarrow X_{i'}$, s.t. $F(f_{ii'})(x_i) = x_{i'}$, as $X_i, X_{i'}$ Galois.
Claim: F is isomorphic to the functor: $F': X \longmapsto \operatorname{colim}_{i \in I} Mor_{\mathcal{C}}(X_i, X)$.
$H = \lim_{i \in I} Aut(X_i)$. ($H^{opp} = Aut(F')$. ?)
$$\begin{tikzcd} \lim_{i \in I} Aut(X_i) \arrow[d] \\ Aut(X_i) \end{tikzcd}$$
So $Aut(F)$ acts trans. on Galois obj., but every conn. obj are dom. by.

Lemma ∀ X, Y conn. ~~(I don't think we need this condition)~~, $\exists\, Z \xrightarrow{f} X\times Y$, w/ Z Galois, Moreover, fix $z \in F(Z)$, $\forall\, x \in F(x), y \in F(Y)$, $\exists$ f_{xy} s.t. $F(f_{xy})(z) = (x,y)$.

e.g. ~~$\mathcal{C}$ of k category of~~ $\mathcal{C} = \{\text{finite sep. alg.}/K\}^{opp}$.
$F: L \longrightarrow \mathrm{Hom}_K(L, \bar{K})$. connected ones are fields!

Prop. $(\mathcal{C}, F)$ Galois cat. Then $F: \mathcal{C} \longrightarrow \mathrm{Aut}(F)\text{-Sets}$ is eq.

pf. ① F is faithful, as it's faithful on connected ones.

② F is full: let $X, Y \in \mathrm{Ob}(\mathcal{C})$, $s: F(X) \longrightarrow F(Y)$ compatible w/ $\mathrm{Aut}(F)$-action, then $\Gamma_s \subseteq F(X)\times F(Y) = F(X\times Y)$. So $\exists\, Z \subseteq X\times Y$, s.t. $\Gamma_s = F(Z)$, $Z \longrightarrow X$ is isom. So $p_2 \circ p_1^{-1}: X \longrightarrow Y$ gives rise to $s: F(X) \to F(Y)$.

③ essentially surjective: Enough to construct X w/ $F(X) \cong \mathrm{Aut}(F)/H$, $H \subseteq \mathrm{Aut}(X)$ open. Can find Y Galois w/ $U = \ker(\mathrm{Aut}(F) \to \mathrm{Aut}(F(Y)))$. contained in H. Then by fullyfaithfulness

$$\begin{tikzcd}
\mathrm{Aut}(Y) \arrow[d, phantom, sloped, "\subseteq"] & \cong \mathrm{Aut}_{\mathrm{Aut}(F)\text{-Sets}}(F(Y)) \cong \mathrm{Aut}_{\mathrm{Aut}(F)\text{-Sets}}(\mathrm{Aut}(F)/U) \cong & (\mathrm{Aut}(F)/U)^{opp} \arrow[d, phantom, sloped, "\subseteq"] \\
H' \arrow[rr, leftrightarrow, dashed] & & (H/U)^{opp}
\end{tikzcd}$$

$$X \triangleq \mathrm{Coeq}\Big(\begin{tikzcd} Y \arrow[r, shift left=2, "h_1'"] \arrow[r, shift right=2, "h_r'"'] & Y \end{tikzcd}\Big). \quad H' = \{h_1', \dots, h_r'\}.$$

So $F(X) = F(Y)/H' = \mathrm{Aut}(F)/H$.

Addendum Let $(\mathcal{C}, F)$, $(\mathcal{C}', F')$ be two Galois cats, $H: \mathcal{C} \longrightarrow \mathcal{C}'$ exact.
Then: ① $\exists$ isom. $t: F \longrightarrow F' \circ H$.
② choice of t determines $h: \mathrm{Aut}(F') \longrightarrow \mathrm{Aut}(F)$, well-defined up to inner aut of $\mathrm{Aut}(F)$.
③

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] \arrow[d, "\wr"] & \mathcal{C}' \arrow[d, "\wr"] \\
\text{Finite-}G\text{-Sets} \arrow[r, "h"] & \text{F.-}G'\text{-Sets}
\end{tikzcd}$$

is 2-commutative (via t).

Defn. unramified morphisms:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & C \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & C/I
\end{tikzcd}$$

$I^2 = 0$.

- f is formally unramified if ! lift $B \to C$
- f is formally smooth if $\exists$ lift $B \to C$
- f is formally Étale if $\exists!$ lift $B \to C$.

Lemma $A \xrightarrow{f} B$ is formally unramified iff $\Omega_{B/A} = 0$.

pf. let $C_{univ} = B \otimes_A B / J^2$, $I_{univ} = J/J^2$.
let $B \underset{\sigma_2}{\overset{\sigma_1}{\rightrightarrows}} C_{univ}$ $\sigma_1(b) = b\otimes 1$, $\sigma_2(b) = 1 \otimes b$. If f is f.unr. then $b\otimes 1 = 1\otimes b$ in C_{univ}. As $(\Omega_{B/A}, d) = (J/J^2, \sigma_1 - \sigma_2)$
Hence $\Omega_{B/A} = 0$. Conversely, if $\tau_1, \tau_2: B \longrightarrow C$, 2 lifts, $B \otimes_A B \xrightarrow{\tau_1 \otimes \tau_2} C$, factor thru C_{univ}, now if $\Omega_{B/A} = 0$.
We have $I_{univ} = 0$, so factor thru $C_{univ}/I_{univ} = B$, hence $\tau_1 = \tau_2$.

Cor. "formally unramified" is local on Source and target.

Defn. $f: X \longrightarrow Y$ is unramified if f is l.f.t. and formally unramified.

Lemma. $f: X \longrightarrow Y$ lft. Then f is unramified iff $\Delta: X \longrightarrow X \times_Y X$ is an open immersion.

pf: In general, let $W = \bigcup_{\substack{V \subseteq X \\ \text{open affine}}} V \times_Y V \overset{\text{open}}{\subseteq} X \times_Y X$, $X \xrightarrow{\Delta} W$ is a closed ~~[illegible]~~ immersion. Let $\mathcal{I}$ be the ideal sheaf, then $\mathcal{I}/\mathcal{I}^2 = \Omega_{X/Y}$. By Nakayama, $V(\mathcal{I}) \subseteq W$ open iff $\mathcal{I}/\mathcal{I}^2 = 0$ (exercise).

Cor. Let $f: X \longrightarrow Y$ be lft, then f is unramified iff for all $x \in X$, $y \overset{\circ}{=} f(x)$. $\mathfrak{m}_x = \mathfrak{m}_y \mathcal{O}_{x,X}$ and $k(x)/k(y)$ is sep. finite.

Defn. $f: X \longrightarrow Y$ is étale if it's l.f.p., flat and unramified.

e.g. $B = (A[x]/(f))_{(f')}$, $A \longrightarrow B$ is étale for monic f.

Thm* Locally, every étale morphism is given by a standard one as above

Lemma Composition of étale is étale, stable under base change.

Prop. Let
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$
if g is unramified, h is étale, then f is étale.

pf.
$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d, "f \times \mathrm{id}_Y"] \arrow[dl, phantom, "\lrcorner"] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd} \qquad \begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "\pi_2"'] & X \arrow[d] \arrow[dl, phantom, "\lrcorner"] \\ Y \arrow[r] & Z \end{tikzcd}$$
$f = \pi_2 \circ \Gamma_f$.

Prop. $f: X \longrightarrow Y$, morphism of schemes, TFAE:

① f is étale

② f is smooth of rel. dim 0.

③ f is flat, lfp, fibre étale.

④ f is lfp, and formally étale.

e.g. $Y = \operatorname{Spec}(k)$, then $X \longrightarrow Y$ is étale iff $\exists\, I$ set, $X \simeq \coprod_{i \in I} \operatorname{Spec} k_i$ where k_i/k finite separable.

Thm* $A \longrightarrow B$ étale $\Longleftrightarrow$ $B = A[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$, s.t. $\det\left(\frac{\partial f_i}{\partial x_j}\right)$ invertible in B.

Property $A \longrightarrow B$ finite étale $\Longleftrightarrow$ B finite, locally free A-mod., and $Q: B \times B \longrightarrow A$, $(b_1, b_2) \longmapsto \mathrm{Tr}_{B/A}(b_1 b_2)$ is nondeg.

Notation. X scheme FÉt_X the cat. of $Y \xrightarrow{f} X$ finite étale, morphisms ~~are~~ as X-schemes.

e.g. $X = \text{Spec}(A)$, then $\text{FÉt}_X = \{\text{sep. finite } A\text{-alg.}\}^{op}$.

Lemma. FÉt_X has all finite lims and colims, and $\forall X' \to X$,

$$\text{FÉt}_X \longrightarrow \text{FÉt}_{X'}, \quad Y/X \longmapsto X' \times_X Y/X'$$ is exact.

pf. check $\exists$ final obj. and $\exists$ fibre products and comm. w/ base change.

check for colimits, enough to check finite coproduct + coequalizers.

Coproduct is just disjoint unions.

$Y_1 \overset{a}{\underset{b}{\rightrightarrows}} Y_2$ in FÉt_X note $Y_i = \underline{\text{Spec}}_X(f_{i,*}\mathcal{O}_{Y_i})$.

$$f_{1,*}(\mathcal{O}_{Y_1}) \overset{a^\#}{\underset{b^\#}{\leftleftarrows}} f_{2,*}(\mathcal{O}_{Y_2}) \longleftarrow \mathcal{A} = \ker(a^\# - b^\#)$$

$\mathcal{A}$ is a q-coh. sheaf of $\mathcal{O}_X$-alg.

Claim: $\underline{\text{Spec}}_X(\mathcal{A}) \longrightarrow X$ is in FÉt_X and is coeq. of a and b.

(① If $Y \xrightarrow{g} X \ni x$, finite unr. $\exists (U, u) \longrightarrow (X, x)$ s.t.

$Y \times_X U = \coprod_{j=1,\dots,m} V_j$ and $V_j \to U$ closed immersion for all j

and ② étale morphism is open).

(*) So étale locally on X, $Y_i = \coprod_{j=1,\dots,m_i} X$.

(**) for $Y_i = \coprod_{j=1,\dots,m_i} X$, then any X-morphism $Y_1 \to Y_2$ comes from a map $\{1,\dots,m_1\} \longrightarrow \{1,\dots,m_2\}$.

(***) Descent theory says it's enough to check for an étale covering. □

~~Defn. A geometric~~ Let $\bar{x}$ be a geometric on X.

$$F_{\bar x} : \text{FÉt}_X \longrightarrow \text{FÉt}_{\bar x} \longrightarrow \text{Sets}$$

$$Y/X \longmapsto F_{\bar x}(Y) = Y_{\bar x} = \left\{ \bar y \;\middle|\; \begin{tikzcd} & Y \arrow[d] \\ \bar x \arrow[ur, "\bar y"] \arrow[r, "\bar x"'] & X \end{tikzcd} \right\}$$

Thm & Defn. If X is a connected scheme, then $(\text{FÉt}_X, F_{\bar x})$ is Galois. And we call $\text{Aut}(F_{\bar x}) = \pi_1(X, \bar x)$.

pf. already know $\exists$ finite (co)lims and $F_{\bar x}$ exact.

Fact: A finite morphism is closed

Fact: An étale morphism is open

$\Big\} \Rightarrow$ by considering degree decomposition into finitely many conn. components.

Lastly $F_{\bar x}$ reflects isoms (by considering degree).

Lemma. Let $f: X' \to X$ be a morphism of conn. schemes. Let $\bar x'$ be a geom. pt of X', set $\bar x = f(\bar x')$. Then we get a canonical continuous homom. $f_* : \pi_1(X', \bar x') \longrightarrow \pi_1(X, x)$ s.t.

$$\begin{tikzcd} \text{FÉt}_X \arrow[r, "\text{base change}"] \arrow[d, "F_{\bar x}"', "\wr"] & \text{FÉt}_{X'} \arrow[d, "F_{\bar x'}"', "\wr"] \\ \text{Finite } \pi\text{-Sets} \arrow[r, "f_*"] & \text{Finite } \pi'\text{-Sets} \end{tikzcd}$$

Change of base pts. If $\bar x_1, \bar x_2$ are geom. pts of conn. X.

$\pi_1(X, \bar x_1) \cong \pi_1(X, \bar x_2)$ well-defined up to inner automorphism (as 2 functors on π-Sets are essentially the same).

e.g. $\pi_1(\mathrm{Spec}(\mathbb{C}((t))) \overset{?}{=} \hat{\mathbb{Z}}$ (✓)

pf. $\mathbb{C}((t))$ is a local field, ~~tak~~ so that's ~~soluble~~ soluable, hence suffices to find cyclic extn of degree p. $\mathbb{C}((t))/(\mathbb{C}((t)))^p \cong C_p$, generated by t. Hence the result follows.

Notation Given scheme X lft / $\mathbb{C}$, denote X^{an} the usually top. space whose underlying set of pts is $X(\mathbb{C})$.

Properties
- If $f: X \to Y$ is a morphism of schemes lft/$\mathbb{C}$, then ~~f^{an}~~ $f^{an}: X^{an} \longrightarrow Y^{an}$ is continuous.
- If f is an open/closed immersion, then f^{an} is an open/closed immersion.
- If $X = \mathbb{A}^n_{\mathbb{C}} \Rightarrow X^{an} = \mathbb{C}^{an}$ w/ Euclidean top.

Lemma. If $f: X \longrightarrow Y$ is a proper morphism of schemes lft/$\mathbb{C}$, then ~~f^{an}~~ $f^{an}: X^{an} \longrightarrow Y^{an}$ is proper ($\overset{\mathrm{def}}{=\!=}$ f closed + fibres cpt).

pf: · Reduce to projective case by Chow's lemma.
 · Reduce to $\mathbb{P}^n_Y \longrightarrow Y$.

Lemma. If $f: X \to Y$ is a morphism of schemes lft/$\mathbb{C}$ and f is étale at some $x \in X(\mathbb{C})$ then f^{an} is a local isom at x:

$$\exists\, x \in U \underset{\text{open}}{\subseteq} X^{an},\ f(x) \in V \underset{\text{open}}{\subseteq} Y^{an},\ \text{s.t.}$$
$$f^{an}|_U : U \longrightarrow V \text{ is a homomorphism.}$$

pf. shrink X, Y so they are affine and f is étale.

— first proof: reduce to $Y = \mathbb{A}^n_{\mathbb{C}}$; to do this $\exists$:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & W \arrow[d] \\ Y \arrow[r, hook] & \mathbb{A}^n_{\mathbb{C}} \end{tikzcd}$$

s.t. $X = Y \times_{\mathbb{A}^n_{\mathbb{C}}} W$, W affine, $W \xrightarrow{\text{étale}} \mathbb{A}^n$

Alg. lemma: A ring, $A/I \longrightarrow \bar{B}$ étale, $\exists\, A \to B$ étale, s.t. $\bar{B} \cong B/IA$.

pf. Write $\bar{B} = A/I[x_1, \cdots, x_n]/(\bar{f}_1, \cdots, \bar{f}_n)$, w/ $\bar{\Delta} = \det(\frac{\partial \bar{f}_i}{\partial x_j})$ invertible in $\bar{B}$, set $B = A[x_1, \cdots, x_n]/(f_1, \cdots, f_n)[\frac{1}{\det(\frac{\partial f_i}{\partial x_j})}]$.

Then we have $W = \mathrm{Spec}\, \mathbb{C}[x_1, \cdots, x_n, z_1, \cdots, z_m]/(F_1(z,x), \cdots, F_m(z,x))$ and $\det(\frac{\partial F_i}{\partial z_j})$ is invertible at x, then use implicit function.

— Second pf: use structure thm: let $A \to B$ be a ring map, étale at $q \subseteq B$, $\exists\, g \in B$, $g \notin q$, s.t. $B_g \cong (A[z]/(P))_{(Q)}$, where $P, Q \in A[z]$, P monic, $\frac{dP}{dz}$ invertible in B_g.

Suppose $P = z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z]$, and α is a simple root. Then $\exists\, \varepsilon > 0$, s.t. for $|b_i| < \varepsilon$, $i = 1, \cdots, n$ the poly.

$$P_{b_1, \cdots, b_n}(z) = z^n + (a_1 + b_1) z^{n-1} + \cdots + (a_n + b_n) \text{ has a simple root.}$$

$\alpha(b_1, \cdots, b_n)$ depend continuously on b_i's and converging to 0 while b_i's $\to \infty$. (Newton's Method).

Cor. Given X lft/$\mathbb{C}$, $\exists$ functor: $X \longrightarrow X^{an}$, $\mathrm{F\acute{E}t}_X \longrightarrow \{\text{fnt cover of } X^{an}\}$.

Thm (Riemann's Existence Thm) This functor is an equivalence.

pf only for X smooth + proj:

- essentially full; $M \xrightarrow{\pi} X^{an}$ finite covering, then M has a canonical structure as cplx mfld.

  Let $\mathcal{L}$ be a positive line bdle on X^{an}, $\pi^*\mathcal{L}$ would be positive, so $M \hookrightarrow \mathbb{P}^m_{\mathbb{C}}$, hence algebraic by Chow's Thm.
- faithful is trivial. (graph ~~has~~ would be reduced).
- fullness: apply Chow's Thm to the graph. (or by GAGA).

Cor. $\pi_1^{top}(X,x)^\wedge \cong \pi_1^{alg.}(X,x)$.

Rmk we do need finite, consider $(\mathbb{A}^1_{\mathbb{C}})^{an} = \mathbb{C} \xrightarrow{\exp} \mathbb{G}_{m,\mathbb{C}}^{an} = \mathbb{C}^*$ isn't analytification of any f. (essential singularity at ∞!)

~~Chow~~

Prop. (Structure of smooth morph's) $X \xrightarrow{f} S$ morphism of schemes.

TFAE. (a) f is smooth

(b) $\forall x \in X$, $\exists$ $X \overset{open}{\supseteq} U$

$$\begin{tikzcd} U \arrow[r, "\text{étale}"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{A}^n_S \arrow[dl] \\ S & \end{tikzcd}$$

2 more examples.

If X smooth, proj., genus $g/\mathbb{C}$, $\pi_1(X) = F_{2g}/([a_1,b_1]\cdots[a_g,b_g])^\wedge$.

$A/\mathbb{C}$ abel. vty, dim. g, $\pi_1(A) \cong (\hat{\mathbb{Z}})^{2g}$

Fund. gp of normal ~~gp~~ schemes.

Lemma Let A be a normal ~~Nthr~~ Nthr. domain, w/ f.f. K, let L/K finite sep. extn, then $\bar{A}$ in L is fin /A.

pf. choose $\{e_i\}$ basis of L/K, w/ $e_i \in \bar{A}$, then $\{e_i\} \subseteq B \subseteq B^* \subseteq \{e_i\}^*$. $B \subseteq B^*$ as $Tr_{L/K}(B) \subseteq A$, where we have to use A is normal.

~~Cor. let~~ Nonexample: when A's not normal? ..

Cor. Let X be a normal Nthr. integral scheme, w/ ~~Fr(X)~~ = f.f. K, L/K sep. There $\exists$ finite dominant $Y \twoheadrightarrow X$, s.t. Y normal integral, f.f. L.

Defn. The Y of cor. is called the normalization of X in L.

Defn. We say X is unramified in L, if $Y \to X$ is unramified.

Lemma 1. In ~~this~~ situation of cor., we have $Y \to X$ unramified $\Leftrightarrow$ étale.

pf. Fact: Let A be a local domain that is integrally closed in $Fr(A) = K$. Denote $\mathfrak{m}$, max. ideal in A. L/K finite, separable, assume $B = \bar{A}$ is f.g. as A-module, $B/\mathfrak{m}B$ is separable $A/\mathfrak{m}A$ algebra. Then B is free of rk $[L:K]$ as an A-module.

Prop. Let X be a Nthr. normal integral scheme w/ f.f K. Then

$$\mathrm{Gal}(K^{sep}/K) = \pi_1(\mathrm{Spec}(K), \mathrm{Spec}(\bar{K})) \longrightarrow \pi_1(X, \mathrm{Spec}(\bar{K})).$$

is surj. and identifies $\pi_1(X, \mathrm{Spec}(\bar{K}))$ w/ the quotient,

$$\mathrm{Gal}(K^{sep}/K) \longrightarrow \mathrm{Gal}(M/K)$$ where $M =$ composition of all that $K \leq L \leq K^{sep}$, s.t. X is unramified in L, M/K is Galois

Fact Lemma If $f: Y \to X$ is étale (smooth), then X has $P \Rightarrow Y$ has P, where P could be normal, regular, CM, etc. Converse will hold if $Y \to X$ is surj.

Key lemma If $Y \to X$ is finite étale, Y connected, then Y is normal integral and of course the normalization of X in $k(Y)$.

remains to show X unramified in $L, L' \Rightarrow X$ unramified in LL'. but then say

$$\begin{tikzcd} B \subseteq L \arrow[r, dash] \arrow[d, dash] & LL' \arrow[d, dash] \\ A \subseteq K \arrow[r, dash] & L' \supseteq B' \end{tikzcd}$$

, then $\mathrm{Spec}(B \otimes_A B')$ has a component which have f.f. LL'. And this component of course is étale/A (by structure thm of étale, maybe?).

Eg.& Cor. $\pi_1(\mathrm{Spec}(\mathbb{Z})) = \{1\}$ $\quad \pi_1(\mathbb{A}^1_{\mathbb{C}}) = \{1\}$. $\quad \pi_1(\mathbb{A}^1_{\bar{\mathbb{F}}_p}) \neq \{1\}$.

(Artin-Schreier) $\Rightarrow \mathrm{Hom}(\pi_1(\mathbb{A}^1_{\bar{\mathbb{F}}_p}), \mathbb{Z}/p\mathbb{Z}) \cong \Gamma(\mathcal{O}_{\mathbb{A}^1_{\bar{\mathbb{F}}_p}})/(f^p - f)$.

Action of Galois gps on fund. gps

Prop. Let k be a perfect field. Let X be a q-cpt and q-sep. scheme /k. Assume $X_{\bar{k}}$ connected (X geom. conn.). Pick $\bar{\xi} \in X_{\bar{k}}$. Then $\exists$ short exact sequence:

$$1 \longrightarrow \underbrace{\pi_1(X_{\bar{k}}, \bar{\xi})}_{\text{geom. fund. gp of } X} \longrightarrow \pi_1(X, \bar{\xi}) \longrightarrow \pi_1(\underbrace{\mathrm{Spec}(k)}_{\mathrm{Gal}(\bar{k}^{sep}/k)}, \bar{\xi}) \longrightarrow 1$$

Method of proof is to prove corresponding results for

$$\mathrm{FÉt}_{X_{\bar{k}}} \longleftarrow \mathrm{FÉt}_X \longleftarrow \mathrm{FÉt}_{\mathrm{Spec}(k)}$$

Rmk this propsition can be viewed as $X_{\bar{k}} \longrightarrow X \longrightarrow \mathrm{Spec}(k)$ fibration, and $\mathrm{Spec}(k)$ is a $K(\pi_1(\bar{k}/k), 1)$. Amazing !!!!!!!!!!

E.g. $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Fact: $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0,1,\infty\}) \xrightarrow{\sim} \pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\})$

So $\quad 1 \longrightarrow \hat{F}_2 \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1$.

gives rise to continuous gp homomorphism:

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}(\hat{F}_2) = \frac{\mathrm{Aut}(\hat{F}_2)}{\mathrm{Inn}(\hat{F}_2)}.$$

pf of the proposition

	$G \xleftarrow{h} G' \xleftarrow{h'} G''$	$\mathcal{C} \xrightarrow{H} \mathcal{C}' \xrightarrow{H'} \mathcal{C}''$
	h surj.	(A) H fully faithful $\Leftrightarrow$ Galois obj. (conn.) sent to Galois obj. (conn.)
$(\overset{?}{\Rightarrow})$	h inj.	(B) $\forall X'' \in \mathcal{C}''$ connected $\exists\ H'(X') \xleftarrow{\text{mono}} Y'' \xrightarrow{\text{epi}} X''$.
	$h \circ h'$ trivial	(C) $H' \circ H(X) \cong \coprod *_{\mathcal{C}''}$ for all X in $\mathcal{C}$.
	$\mathrm{Im}(h')$ normal.	(D) $\forall X' \in \mathcal{C}'$ conn., if $\exists *_{\mathcal{C}''} \to H'(X')$ then $H'(X') \simeq \coprod *_{\mathcal{C}''}$.
	h surj. + kernel h is smallest normal closed subgp cont. $\mathrm{Im}(h')$	(E) H fully faithful + essential image of H is exactly those X', s.t. $H'(X') \simeq \coprod *_{\mathcal{C}''}$.

Fact (E) $\Leftrightarrow$ (A) + (C) + ($H'(X') \cong \coprod *_{\mathcal{C}''} \Rightarrow \exists X \in \mathcal{C}$ and an epi $\begin{matrix} X' \\ \uparrow \\ H(X) \end{matrix}$).
because w/ (A) (C) assumed, third condition becomes $\overline{h'(G'')} \supseteq \ker(h)$.

Now apply this to $\text{FÉt}_k \longrightarrow \text{FÉt}_X \longrightarrow \text{FÉt}_{X_{\bar k}}$ w/ X/k qc + qsep + geom. conn. (and assume k is perfect).

For (A) have to show k'/k finite sep. field extn, then $X_{\bar k} \twoheadrightarrow X_{k'}$, and X geom. conn., so $X_{k'}$ is connected.

For (B) For $Z \longrightarrow X_{\bar k}$ finite étale, $\exists\ \bar k / k' / k$ and $Y \longrightarrow X_{k'}$ finite étale, s.t. $Z = Y \times_{k'} \bar k$, ~~Then~~ Then $Y \longrightarrow X$ is finite étale and Z is a connected component of $Y \times_k \bar k$.

Lemma. If $X = \lim_{i \in I} X_i$, X_i is qc + qsep $\forall i$, then
$\text{FÉt}_X = \operatorname{colim}_i \text{FÉt}_{X_i}$ and if X_i connected $\forall i >> 0$
then $\pi_1(X) = \lim \pi_1(X_i)$

For (C) it's trivial

For (D) (?) Suppose $U \longrightarrow X$ finite étale connected, $\exists\ s : X_{\bar k} \longrightarrow U_{\bar k}$,
consider $\overline{T} = \bigcup_{\sigma \in Gal(\bar k/k)} s^{\sigma}(X_{\bar k}) \subseteq U_{\bar k}$ which is $G(\bar k/k)$-inv
and open + closed $\Rightarrow \overline{T}$ is inverse image of $T \underset{\text{open+closed}}{\subseteq} U$.
$\underset{(U \text{ conn.})}{\Longrightarrow} T = U$, $\overline{T} = \overline{U}$.

For (E) $U \longrightarrow X$ finite étale & $U_{\bar k} = \coprod X_{\bar k}$. Then $\exists\ U_{k'} = \coprod X_{k'}$
So $U \twoheadleftarrow U_{k'} = \coprod X_{k'} = X \times_k \coprod k'$.

Rmk if k not perfect, we've proved SES: $1 \longrightarrow \pi_1(X_{k^{sep}}) \longrightarrow \pi_1(X)$
$\longrightarrow G(k^{sep}/k) \longrightarrow 1$.

Ex. $X = \mathbb{G}_{m,k} = \mathbb{A}^1_k \setminus \{0\}$ char$(k) = 0$.
$X_{\bar k} = \mathbb{G}_{m,\bar k}$ $\quad X_n = \mathbb{G}_{m,\bar k} \xrightarrow{(\ )^n} X_{\bar k}$.
$\pi_1(X_{\bar k}) = \lim_n \mu_n(\bar k) \cong \hat{\mathbb{Z}}(1)$.
$0 \longrightarrow \pi_1(X_{\bar k}) \longrightarrow \pi_1(X) \longrightarrow Gal(\bar k/k) \longrightarrow 1$
$Gal(\bar k/k) \circlearrowright \pi_1(X_{\bar k})$ by the action on $\mu_n(\bar k)$.

Ex. $X \rightleftarrows \mathrm{Spec}(k)$ elliptic curve, $\mathrm{char}(k)=0$, with section O

$\pi_1(X_{\bar{k}}) \cong \hat{\mathbb{Z}}^{\oplus 2} \longleftarrow$ multiplication by n gives cofinal.

$$\begin{tikzcd} 0 \arrow[r] & \hat{\mathbb{Z}}^{\oplus 2} \arrow[r] & \pi_1(X) \arrow[r, shift right] & \mathrm{Gal}(\bar{k}/k) \arrow[l, shift right, "O"'] \end{tikzcd}$$

$\mathrm{Gal}(\bar{k}/k) \circlearrowright \hat{\mathbb{Z}}^{\oplus 2}$, $\mathrm{Gal}(\bar{k}/k) \longrightarrow \prod_{\ell} GL_2(\mathbb{Z}_\ell)$.

Suppose A is a dvr. $K=\mathrm{Fr}(A)$, L/K sep. finite $B=\bar{A}$ in L.
B finite $/A$.

Fact: ~~B~~ B is a dedakind domain w/ finite max ideals $m_1, \cdots, m_n$.

$A \leq B_{m_i}$ extn of dvr $\rightsquigarrow$ e_i ramification index

$f_i = [k(m_i):k_A]$.

$[L:K] = \sum_i e_i f_i$

If A is complete or henselian, then $n=1$.

Assume L/K Galois, $G=\mathrm{Gal}(L/K)$. choose $m=m_1$.

$\{1\} \subseteq P \subseteq I \subseteq D \subseteq G$.

$\sigma \equiv id \mod m^2$ $\sigma \equiv id \mod m$ $\sigma(m)=m$.

$[G:D]=n$. $k(m)/k(A)$ is normal, not nec. sep.

$I \lhd D$ and $D/I \xrightarrow{\sim} \mathrm{Aut}(k(m)/k_A)$.

$P \lhd D$ and $P = \begin{cases} \{1\} & \text{if } \mathrm{char}(k_A)=0 \\ p\text{-gp} & \text{if } \mathrm{char}(k_A)=p. \end{cases}$

I/P is cyclic of order prime-to-p-part of e.

tame inertia of m, (can. isom. $I/P \xrightarrow{\sim} \mu_e(k(m))$)

Now if A is henselian, then $P \subseteq I \subseteq D = \mathrm{Gal}(K^{sep}/K)$, by passing to the limit.

So P and I are normal in G and

(1) $G/I \simeq \mathrm{Gal}(k_A^{sep}/k_A)$

(2) $I_t = I/P \underset{(*)}{\cong} \prod_{\ell \neq \mathrm{char}(k_A)} \mathbb{Z}_\ell$.

(3) $1 \longrightarrow I_t \longrightarrow G/P \longrightarrow \mathrm{Gal}(k_A^{sep}/k_A) \longrightarrow 1$ gives ~~Gal(k_A~~
$G(k_A^{sep}/k_A) \circlearrowright \prod_{\ell \neq p} \mathbb{Z}_\ell$ via cyclotomic character.

(4) $P = \{1\}$ if $\mathrm{char}(k_A)=0$.

$(*)$ is noncanonical, $I_t = \mu_e(k_A^{sep})$.

$\theta: I/P \longrightarrow \mu_e(k(m))$.

$\sigma(\pi) = \widetilde{\theta(\sigma)} \cdot \pi \mod \pi^2 B_m$, $\pi \in B_m$ a uniformizer.

$\widetilde{\theta(\sigma)} \in B_m$ is any lift of $\theta(\sigma) \in k(m)$.

$\pi^e = u \cdot \pi_A$, so $\theta(\sigma)$ will be root of unity in $k(m)$.

Ex. $A = k[[t]]$ w/ $k=\bar{k}$, $\mathrm{char}(k)=0$. then $I=D=G$. $P=\{1\}$

$I/P \simeq \hat{\mathbb{Z}}$.

Ex. $\bar{X}$ sm. curve $/k=\bar{k}$, $\mathrm{char}(k)=0$. Let $X = \bar{X} \setminus \{x_1, \cdots, x_m\}$

Apply w/ $A_i = \hat{\mathcal{O}}_{X,x_i}$ or $\mathcal{O}^h_{X,x_i}$. we get $a_i: \mathrm{Spec}(ff(A_i)) \to X$

Hence $\hat{\mathbb{Z}} \simeq \pi_1(\mathrm{Spec}(f.f.(A_i)) \xrightarrow{\pi_1(a_i)} X$.

well-defined up to inner conjugation.

Lemma. the kernel of surj. map $\pi_1(X) \twoheadrightarrow \pi_1(\bar{X})$ is the smallest closed normal subgp of $\pi_1(X)$ containing the images of $\pi_1(a_i)$

pf. consider $F\acute{E}t_{\bar{X}} \to F\acute{E}t_X \to F\acute{E}t_{A_i}$.

for surj. we use the knowledge about fund. gp of normal int'l scheme

Cor. $\exists$ SES $\hat{\mathbb{Z}}^{\oplus m} \longrightarrow \pi_1(X)^{ab} \longrightarrow \pi_1(\bar{X})^{ab} \longrightarrow 0$.

Ex. $\bar{X} = \mathbb{A}^1_k$ and $X = \mathbb{G}_{m,k}$ so $m=1$, $x_1 = 0$.

$$\begin{tikzcd}
\text{inertia at } 0 \arrow[r, "\sim"] \arrow[rr, bend right, "-1"'] & \pi_1(X) & \text{inertia at } \infty \arrow[l, "\sim"']
\end{tikzcd}$$

Thm. Let $f: X \to Y$ be a universal homeomorphism of schemes. Then the base change functor: $\acute{E}t/Y \longrightarrow \acute{E}t/X$ is an eq.
Same for $F\acute{E}t_Y \to F\acute{E}t_X$, and if X, Y conn., $\pi_1(X) \xrightarrow{\sim} \pi_1(Y)$.

Defn. $f: X \to Y$ is a universal homeomorphism if $\forall\ Y' \to Y$,
$X \times_Y Y' \longrightarrow Y'$ is a homeomorphism.

Fact f univ. homeo. $\Longleftrightarrow$ f integral, surj., univ. inj.

Ex. (A) thickenings. $X \xrightarrow{i} X'$ is called a thickening iff i is a closed immersion and bij.

(B) purely inseparable extn. of fields.

(C) $\mathrm{Spec}(A) \xrightarrow{f} \mathrm{Spec}(B)$ and $\exists\ g: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, s.t.

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[rr, "f"] \arrow[dr] & & \mathrm{Spec}(B) \arrow[dl] \\
& \mathrm{Spec}\,\mathbb{F}_p &
\end{tikzcd}$$

$g \circ f = (A \to A,\ a \mapsto a^q)$.
$f \circ g = (B \to B,\ b \mapsto b^q)$.

subex. $A = \mathbb{F}_p[x_1, \dots, x_n]/I$ $\quad B = \mathbb{F}_p[x_1, \dots, x_n]/I^{3?}$

Thm. $f: X \to \mathrm{Spec}\,A$ proper. If $(A, \mathfrak{m}, \kappa)$ henselian local ring & $X_0 = X \times_A \kappa$, then $F\acute{E}t_X \xrightarrow{\sim} F\acute{E}t_{X_0}$

Rmk (1) $\pi_1(X) = \pi_1(X_0)$.
(2) $\eta \in \mathrm{Spec}(A)$ is a generic pt.

$$\begin{tikzcd}
\pi_1(X_{\bar{\eta}}) \arrow[r] \arrow[rrr, bend right, "\text{Specialization map}"'] & \pi_1(X_\eta) \arrow[r] & \pi_1(X) \arrow[r, equal, "\text{Thm}"] & \pi_1(X_0)
\end{tikzcd}$$

sketch of pf: given $Y_0 \to X_0$ finite étale
we can find $Y_n \xrightarrow{f.\acute{e}t.} X_n$ by thm above
algebrization (Grothendieck's existence) $\exists\ \hat{Y} \longrightarrow \hat{X}$. finite
then use lemma:

Lemma $f: Y \to X$ lft w/ X local Noeth. let $X_0 \hookrightarrow X$ closed subscheme w/ inf. nbhd $X_n \hookrightarrow X$. If $Y_n \to X_n$ smooth (étale), $\forall n \geq 0$, then f is smooth (étale) at all pts of $f^{-1}(X$

Now $X \longrightarrow \mathrm{Spec}(A)$ proper & A local $\Rightarrow$ any open nbhd of X_0 is X.
So the $\hat{Y}$ we got is étale over $\hat{X}$.

~~Then~~ ~~we~~ As $\hat{A} = \mathrm{colim}\, A_i$, where $A \to A_i$ of f.t.
$\exists\ i$ and $Y_i \longrightarrow A_i \times_A X$ finite étale, closed fibre Y_0.

write $A_i = A[x_1, \dots, x_n]/(f_1, \dots, f_m) \longrightarrow \hat{A}$, by approximation
$x_i \longmapsto b_i$

we get $(a_i) \in A$ w/ $f_j(a_i) = 0$, $a_i \equiv b_i \mod m\hat{A}$. So
$\exists\ \ell : A_i \longrightarrow A$. Take $Y = A \times_{A_i} Y_i$, we see
$x_i \longmapsto a_i$

$Y \longrightarrow X$ étale, finite (as is base change),
and as $a_i \equiv b_i \mod m\hat{A}$, we see Y has closed fibre Y_a.

And as any henselian ring is a filtered colim of $A = C^h$ where C is a Noeth local ring ess. f.t. $/\mathbb{Z}$ (where approximation for $A \longrightarrow \hat{A}$ holds), we win!

Approximation (Artin... Popescu) Lemma/Defn. (A, m, K) local. The henselization is $A^h = \operatorname{colim}_{A \to B, q} B$ $= \operatorname{colim}_{A \to B, q} B_q$, where colim is over of $A \longrightarrow B$ étale and $q \subseteq B$ prime lying over m, s.t. $K(q) = K(m)$.
The index cat. is filtered.

Defn. A local ring (A, m, K) is henselian if hensel's lemma holds.
i.e., simple root of monic A-coeff poly. lifts.

Lemma A^h is henselian.

Lemma. If A noeth. $\Rightarrow A \underset{f.f.}{\subseteq} A^h \underset{f.f.}{\subseteq} \hat{A} = (A^h)^\wedge$.

Fact If A excellent Noeth. local ring [e.g. local ring of scheme f.t. / (field or $\mathbb{Z}$)], then approximation holds for $A^h \subseteq \hat{A}$.
i.e., $\forall N \geq 1$, $\forall m, n$, $\forall f_1, \dots, f_m \in R[x_1, \dots, x_n]$, $\forall b_1, \dots, b_n \in \hat{A}$
s.t. $f_j(b_i) = 0$ $\exists\ a_i \in R$, s.t. $a_i \equiv b_i \mod m^N \hat{A}$.

Purity of branch locus.

Lemma/Defn Let $f : X \to Y$ be l.f.t. let $x \in X$ w/ $y = f(x)$
TFAE. ① f is q-finite at x
② x is isolated in X_y ($\overset{\text{defn}}{\Longleftrightarrow}$ $\{x\}$ is open in X_y)
③ x is closed in X_y and no $x' \rightsquigarrow x$ in X_y except $x' = x$
④ for some (any) affine opens $\begin{matrix} \operatorname{Spec}(A) \subseteq X \\ \downarrow \qquad \quad \downarrow f \\ \operatorname{Spec}(B) \subseteq Y \end{matrix}$ x corresponds to $\mathfrak{p}$, then $B/\mathfrak{p} \cap B \longrightarrow A/\mathfrak{p}$ is q-finite.

Defn. $f : X \to Y$ is locally quasi-finite if f q-finite at every $x \in X$.
— " — quasi-finite — " if l.q.f. + q.c.

Easy Case $f : X \to Y$ and $x \in X$, Assume
① X, Y locally Noetherian
② f is l.f.t. and quasi-finite at x.
③ f is <u>flat</u>
④ x is not a generic pt of X
⑤ $\forall\ x' \rightsquigarrow x$ w/ $\dim(\mathcal{O}_{X,x'}) = 1$ have f unramified at x'
Then f is étale at x.

Fact — étale locally a q-finite morph. becomes finite.

reduce to: — $A \to B$ finite, flat, A noeth. local ring. If $\dim(A) \geq 2$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is étale over $\mathrm{Spec}(A) \setminus \{m_A\}$.
Then $A \to B$ is étale.

pf. flat, finite $\Rightarrow B = A^{\oplus n}$, $Q: B \times B \to A$, $(b_i, b_j) \mapsto \mathrm{Tr}(b_i b_j)$
$\mathrm{disc}_{B/A} = \det(Q)$ Branch locus cut out by disc ~~[illegible]~~ is codim 1, so we win (as $\dim A \geq 2$).

Difficult case — $f: X \to Y$, $x \in X$, $y = f(x)$. Assume:
① X and Y locally Noeth.
② $\mathcal{O}_{X,x}$ normal
③ $\mathcal{O}_{Y,y}$ regular
④ f is quasi-finite at x.
⑤ $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} \geq 1$
⑥ $\forall\, x' \rightsquigarrow x$, $\dim(\mathcal{O}_{X,x'}) = 1$, ~~then~~ then f unramified at x'.
~~then f unramified at x'.~~ Then f is étale at x.

Alg. problem — $A \subseteq B$ finite extn, w/ A reg. local, $\dim A \geq 2$, B normal. If $A \to B$ is étale at punctured spectrum, then $A \to B$ is étale.

case I — $\dim A = 2$. In this case $A \to B$ will be flat.

Fact 1. — $f: X \to Y$, q-finite, dominant, Y reg., $\dim Y = 2$, X normal, both integral $\Rightarrow f$ is flat.

Fact 2. — $f: X \to Y$, X, Y integral, f dominant, q-finite, Y reg. X CM $\Rightarrow f$ is flat.

case II — $\dim(A) \geq 3$. Pick $f \in m_A \setminus m_A^2$.

$$\begin{tikzcd}
\mathrm{Spec}(A) = X \arrow[r, leftrightarrow] & X_0 = \mathrm{Spec}(A/fA) \\
X \setminus \{m\} = U \arrow[u, phantom, "\cup" description] \arrow[r, leftrightarrow] & U_0 = \mathrm{Spec}(A/fA) \setminus \{m\} \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

By induction, have the result for A/fA. So $\mathrm{F\acute{E}t}_{X_0} \to \mathrm{F\acute{E}t}_{U_0}$ is an eq.
If A is henselian, then we have $\mathrm{F\acute{E}t}_X \cong \mathrm{F\acute{E}t}_{X_0} \cong \mathrm{F\acute{E}t}_{\mathrm{Spec}(k_A)}$.

Goal — $\mathrm{F\acute{E}t}_U \to \mathrm{F\acute{E}t}_{U_0}$ is an eq., when A is henselian (or something)

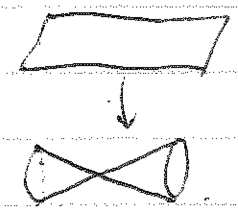
Thm. — Let (A, m) be a local Noeth ring. Let $f \in m$ be a nonzerodiv. Assume $H^1_m(A)$ is finite, $H^2_m(A)$ is annihilated by a power of f. A henselian. Then $\mathrm{F\acute{E}t}_U \longrightarrow \mathrm{F\acute{E}t}_{U_0}$ is fully faithful.

Rmk. — $\mathrm{depth}(A) \geq t \Leftrightarrow H^i_m(A) = 0$, $i = 0, \dots, t-1$.
So if $\dim A \geq 3$ and CM, then assumptions are satisfied.

Cor. — local purity holds for A local, $\dim \geq 3$ and A cplete intersection.
~~(which means~~

Defn — A is C.I. if $\hat{A} = (\text{reg. local})/(\text{reg. seq.})$.

Sharp — Not OK if $\dim A = 2$. e.g. $B = k[x, y]$
$$\begin{tikzcd}
B = k[x,y] \\
A = k[x^2, xy, y^2] \arrow[u, hook]
\end{tikzcd}$$



Local Lefschetz.

$(A, \mathfrak{m}, k)$ Noetherian local ring, $f \in \mathfrak{m}$, nonzerodivisor

$$\begin{tikzcd} \operatorname{Spec}(A)-\{\mathfrak{m}\} = \mathcal{U} \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & \mathcal{U}_0 = \operatorname{Spec}(A/fA)-\{\mathfrak{m}\} \arrow[l, hook'] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] \\ \operatorname{Spec} A = X & X_0 = \operatorname{Spec}(A/fA) \arrow[l, hook'] \end{tikzcd}$$

If $\mathcal{U}, \mathcal{U}_0$ connected

$$\begin{tikzcd} \pi_1(\mathcal{U}) \arrow[d] & \pi_1(\mathcal{U}_0) \arrow[l] \arrow[d] \\ \pi_1(X) & \pi_1(X_0) \arrow[l] \end{tikzcd}$$

Observation If A is strictly henselian, then $\pi_1(X) = \pi_1(X_0) = \{1\}$.

Question: Find condition on A, s.t. $\pi_1(\mathcal{U}_0) \twoheadrightarrow \pi_1(\mathcal{U})$ or
$\pi_1(\mathcal{U}_0) \xrightarrow{\sim} \pi_1(\mathcal{U})$ or $\operatorname{Pic}(\mathcal{U}) \hookrightarrow \operatorname{Pic}(\mathcal{U}_0)$ or $\operatorname{Pic}(\mathcal{U}) \xrightarrow{\sim} \operatorname{Pic}(\mathcal{U}_0)$

Thm A henselian, $H^1_m(A)$ finite, f^N kills $H^2_m(A)$ for some N.
Then $\text{FÉt}_{\mathcal{U}} \longrightarrow \text{FÉt}_{\mathcal{U}_0}$ is fully faithful, i.e., $\pi_1(\mathcal{U}_0) \twoheadrightarrow \pi_1(\mathcal{U})$ is surj.
if $\mathcal{U}, \mathcal{U}_0$ are connected.

Rmk If A strictly henselian, and local purity holds for A/fA, i.e.,
$\pi_1(\mathcal{U}_0) \xrightarrow{\sim} \pi_1(X_0)$, then $\text{FÉt}_{\mathcal{U}} \xrightarrow{\sim} \text{FÉt}_{\mathcal{U}_0}$ (local purity holds for A).

Set $V_n = V \times_{\mathcal{U}} \mathcal{U}_n$, $\mathcal{U}_n = V(f^{n+1}) \subseteq \mathcal{U}$, where $V \to \mathcal{U}$ finite étale.
we want to show $\pi_0(V) \xrightarrow{\sim} \pi_0(V_0)$.

$$\operatorname{Idem} \Gamma(\mathcal{O}_V) \longrightarrow \cdots \longrightarrow \operatorname{Idem} \Gamma(\mathcal{O}_{V_1}) \longrightarrow \operatorname{Idem} \Gamma(\mathcal{O}_{V_0}).$$

So it's enough to show $\Gamma(V, \mathcal{O}_V) \xrightarrow{\sim} \lim \Gamma(V_n, \mathcal{O}_{V_n})$.

Lemma (tag 0BLD) In the setting above, $\exists$ modules H^p and s.e.s.

$$0 \to R^1\varprojlim H^{p-1}(\mathcal{O}_{V_n}) \longrightarrow H^p \longrightarrow \varprojlim H^p(\mathcal{O}_{V_n}) \longrightarrow 0. \quad \text{and}$$
$$0 \longrightarrow H^0(H^p(\mathcal{O}_V)^\wedge) \longrightarrow H^p \longrightarrow T_f H^{p+1}(\mathcal{O}_V) \longrightarrow 0 \quad \text{where}$$

- $H^p(\mathcal{O}_V)^\wedge$ is derived completion of $H^0(\mathcal{O}_V)$ w.r.t. f.

and for any A-module M, we have a s.e.s.

$$0 \longrightarrow R^1\varprojlim M[f^n] \longrightarrow H^0(M^\wedge) \longrightarrow \hat{M} = \lim M/f^nM \longrightarrow 0.$$

- $T_f(M) = f$-Tate module of $M = \varprojlim M[f^n]$.

$$\longrightarrow M[f^3] \xrightarrow{f} M[f^2] \xrightarrow{f} M[f]$$
$$\hat{M} = R\lim(\underset{-1}{M} \xrightarrow{f^n} \underset{0}{M})$$

Take $p=0$. • $H^0 = \varprojlim H^0(\mathcal{O}_{V_n})$.

(?) • $H^0(\mathcal{O}_V)^\wedge$ has bdd f-power torsion (V Noeth.).

$$\Rightarrow H^0(H^0(\mathcal{O}_V)^\wedge) = \widehat{H^0(\mathcal{O}_V)}$$

So we get $0 \to \widehat{H^0(\mathcal{O}_V)} \longrightarrow \varprojlim H^0(\mathcal{O}_{V_n}) \longrightarrow T_f H^1(\mathcal{O}_V) \longrightarrow 0$.

Cor. Now it suffices to show $\forall\ V \to \mathcal{U}$ finite étale,
① $H^0(\mathcal{O}_V)$ is f-adically complete
② f-power torsion on $H^1(\mathcal{O}_V)$ is bdd.

proof of ①: when A is complete

$$\left.\begin{array}{l} H^1_m(A) \text{ finite} \\ V \to \mathcal{U} \text{ finite étale} \end{array}\right\} \Rightarrow H^0(V, \mathcal{O}_V) \text{ finite over } A.$$

$\exists$ exact sequence:

$$0 \longrightarrow H^0_m(A) \longrightarrow A \longrightarrow H^0(U, \mathcal{O}_U) \longrightarrow H^1_m(A) \longrightarrow 0.$$

$$H^i(U, \mathcal{O}_U) \xrightarrow{\delta} H^{i+1}_m(A).$$

proof of ②: $H^1(V, \mathcal{O}_V) \underset{\substack{\uparrow \\ V \to U \text{ finite}}}{=} H^1(U, \pi_* \mathcal{O}_V) \underset{\substack{\uparrow \\ \text{base change}}}{=} H^1(U, \mathcal{E})$.

has bdd f-torsion?

$\forall u \in U$, $\exists u \in U' \subseteq U$, s.t. $\mathcal{E}|_{U'} \overset{\varphi}{\rightleftarrows} \mathcal{O}_U^{\oplus r}|_{U'}$,

$U' = D(g)$.

$\exists \alpha : \mathcal{O}_U^{\oplus r} \longrightarrow \mathcal{E}$, s.t. $\alpha|_{U'} = g^{N_1} \varphi$.

$\beta : \mathcal{E} \longrightarrow \mathcal{O}_U^{\oplus r}$, s.t. $\beta|_{U'} = g^{N_2} \varphi^{-1}$.

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\beta"] \arrow[rr, bend right, "g^{N'}"'] & \mathcal{O}_U^{\oplus r} \arrow[r, "\alpha"] & \mathcal{E} \end{tikzcd}$$

, now $\forall u \in U$, $\exists g \in A$, s.t. $u \in D(g)$ and $g^{N'} f^N \in \mathrm{Ann}(H^1(U, \mathcal{E}))$

Ideal generated by all $g^{N'}$'s is m-primary.

$\Rightarrow m^{\bar{N}} \cdot f^N \subseteq \mathrm{Ann}(H^1(U, \mathcal{E})) \Rightarrow f^{\bar{N}+N} \in \mathrm{Ann}(H^1(U, \mathcal{E}))$.

Upshot: done with Thm in complete case. Then we still have to reduce to noncomplete case.

Specialization of the fundamental gp.

Thm1 $k \subseteq k'$ be an extn of alg. closed fields. X/k proper connected scheme. Then X' is connected and $\pi_1(X) = \pi_1(X')$. $X' = X \times_k k'$.

pf. $H^0(X, \mathcal{O}_X) \otimes_k k' = H^0(X', \mathcal{O}_{X'})$

Thm2 (Specialization of fund. gp) Let $X \longrightarrow S$ be a proper smooth morphism w/ geom. conn. fibres. Let $s \rightsquigarrow s'$ be a specialization of points of s. Then $\exists$ a map sp $\pi_1(X_{\bar{s}}) \longrightarrow \pi_1(X_{\bar{s}'})$. Which is surj. and an isomorphism on prime-p-quotients if $\mathrm{char}(k(s')) = p > 0$.

Rmk. If X is conn. then

$$\begin{tikzcd} \pi_1(X_{\bar{s}}) \arrow[rr, "sp"] \arrow[dr] & \circlearrowleft & \pi_1(X_{\bar{s}'}) \arrow[dl] \\ & \pi_1(X) & \end{tikzcd}$$

Rmk. Let $f : X \longrightarrow S$ be a flat proper morphism w/ geom. conn. and reduced fibres. If S is Noeth. and conn. and $\bar{s} \in S$, then $\exists$ exact sequence. $\pi_1(X_{\bar{s}}) \longrightarrow \pi_1(X) \longrightarrow \pi_1(S) \longrightarrow 1$.

Rmk. sp exists only assuming $X \longrightarrow S$ proper.

proof of Thm2. Step 1. reduction.

① reduce to S Noetherian.

② reduce to $S = \mathrm{Spec}(A)$, A dvr.

③ May assume A is complete dvr w/ alg. closed residue field.

Step 2. Now $s=\eta$, $s'=0$, $S=\mathrm{Spec}(A)$, A as above. We get

$$\begin{tikzcd}
\pi_1(X_{\bar\eta}) \arrow[r] \arrow[rr, bend right=30, "sp"'] & \pi_1(X_\eta) \arrow[r] & \pi_1(X) = \pi_1(X_0)
\end{tikzcd}$$

previous thm.

Step 3. sp is surjective

pf. Let $K = f.f.(A)$, $Y_0 \to X_0$ conn. finite étale. $Y \to X$ be the corresponding finite étale cover.

We have to show $Y_{\bar\eta}$ connected. If not, $\exists\ L/K$ finite, s.t. Y_L disconnected. Let $B \subseteq L$ int'l closure of A in L.

Note $A/m_A = B/m_B$ and B local.

$$\begin{tikzcd}
Y_L \arrow[r, phantom, "\subseteq"] \arrow[d] & Y_B \arrow[r] \arrow[d] & Y \arrow[r] \arrow[dr] & X \arrow[d] \\
\mathrm{Spec}(L) \arrow[r, phantom, "\subseteq"] & \mathrm{Spec}(B) \arrow[rr] & & \mathrm{Spec}(A)
\end{tikzcd}$$

$Y_B \to \mathrm{Spec}(B)$ smooth proper & generic fibre disc.

$\Downarrow$

$Y_0 = Y \times_B B/m_B$ discon.

Lemma. $Z \to T$ is smooth proper. Then fctn $t \mapsto \# \pi_0(Z_{\bar t})$ is loc. constant on T.

Step 4. Claim: Let $\pi_1(X_{\bar\eta}) \xrightarrow{\gamma} G$ continuous finite quotient w/ $\# G$ prime to $\mathrm{char}(A/m_A)$, then γ factors thru sp.

pf. $G \circlearrowright Y' \to X_{\bar\eta}$. We can find $K(\bar\eta)/L/K$, where L/K finite Galois. So we ~~have~~ can replace A by B, and assume $G \circlearrowright Y' \to X_L$. Then we get

$$\begin{tikzcd}
Y' \arrow[r, phantom, "\subseteq"] \arrow[d, "\pi_\eta"'] & Y \arrow[d, "\pi"] & \text{normal} \arrow[l] \\
X_L \arrow[r, phantom, "\subseteq"] & X & \text{finite}
\end{tikzcd}$$

As X is regular (smooth/dvr).

where $Y \to X$ is the normalization of X in $f.f.(Y')$.

By purity of branched locus, if π is branched, then it'll be branched at a codim 1 pt.

Such a point always lies over $\xi \in X_0$ generic point of X_0.

Let $y_1, \cdots, y_n$ be generic pts of Y_0. Then $\pi^{-1}(\{\xi\}) = \{y_1, \cdots, y_n\}$

Now $\# G$ prime to $\mathrm{char}(X_\xi) \Rightarrow$ Ramification of $\mathcal{O}_{Y, y_i}/\mathcal{O}_{X,\xi}$ is tame.

$$\begin{tikzcd}
f.f.(X) \arrow[r, phantom, "\subseteq"] & f.f.(Y) \\
\mathcal{O}_{X,\xi} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subseteq"] & \left(\begin{array}{c}\text{int'l closure of} \\ \mathcal{O}_{X,\xi} \text{ in } f.f.(Y)\end{array}\right) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Galois extn w/ G.

Abhyankar's lemma guarantees we can lower e of this extn after base changing w/ suitable B/A finite as before.

Abhyankar's Lemma (Tag 0BRM).

set up: $A \subseteq B$ extn of dvr's

$$\begin{tikzcd}
f.f.(B) = L \arrow[r, phantom, "\supseteq"] & B \\
f.f.(A) = K \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\supseteq"] & A \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$\pi_A = \text{unit} \cdot \pi_B^e$ $\quad \kappa_B/\kappa_A$ extn of residue field (not nec. finite).

Reminder K'/K is sep. iff $\forall$ finite gen. subextn is separably generated.

$A \subseteq B$. Let K_1/K be a finite sep. extn.

$$\begin{tikzcd}
L \supseteq B \arrow[r, dash] \arrow[d, phantom, "\cup"] & B_1 \subseteq L \otimes_K K_1 \arrow[r, dash] \arrow[d, phantom, "\cup"] & \Pi \text{ fnt sep. extn of } L \\
K \supseteq A \arrow[r, dash] \arrow[r, dash, bend right] & A_1 \subseteq K_1 &
\end{tikzcd}$$

$A_1 = \bar{A}$ in K_1

$B_1 = \bar{B}$ in $L \otimes_K K_1$.

$\mathfrak{m}_1, \cdots, \mathfrak{m}_n$ max. ideals of A_1

$\mathfrak{m}_{i_1}, \cdots, \mathfrak{m}_{i m_i}$ max. ideals of B_1 lying over $\mathfrak{m}_i$.

e_i = ram. index of $A \longrightarrow (A_1)_{\mathfrak{m}_i}$.

e_{ij} = —— " —— $(A_1)_{\mathfrak{m}_i} \longrightarrow (B_1)_{\mathfrak{m}_{ij}}$

$\kappa(\mathfrak{m}_{ij})/\kappa(\mathfrak{m}_i)$.

Abhyankar's Lemma. Assume e prime to $\mathrm{char}(\kappa_A)$ and κ_B/κ_A sep. If $e | e_i$, then $e_{ij} = 1$ for all $j = 1, \cdots, m_i$ and $\kappa(\mathfrak{m}_{ij})/\kappa(\mathfrak{m}_i)$ sep.

Rmk. In fact under some ~~assup~~ hyp. you get $e_{ij} = \frac{e}{\gcd(e, e_i)}$.

End of the proof, the upshot is that we may base change A to $A[\pi_A^{1/|G|}]$ ...

Application. (C.I.) Let $X \subseteq \mathbb{P}^n_k$ be a smooth complete intersection and $k = \bar{k}$. If $\dim X \geq 2$, then $\pi_1(X) = \{1\}$

pf. over $k = \mathbb{C}$ use Lefschetz

$\rightsquigarrow$ get it over any char 0, $k = \bar{k}$.

$\longrightarrow$ lift and apply thm. (surjective part).

(curves). Let X be a sm. proj. curve of genus g / $k = \bar{k}$.

Then $\exists$ surj. $\left(\frac{\langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g \rangle}{[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]} \right)^{\wedge} \longrightarrow \pi_1(X)$.

which induces an isom. on ~~max~~ max. prime-to-p quotients.

(AVs). If A_g is an g-dim'l abelian variety / $k = \bar{k}$.

Then $\pi_1(A) = \begin{cases} \hat{\mathbb{Z}}^{\oplus 2g} & \mathrm{char}(k) = 0. \\ \prod_{\ell \neq p} \mathbb{Z}_\ell^{\oplus 2g} \times \mathbb{Z}_p^f & \text{if } \mathrm{char}(k) = p. \end{cases}$

and f = p-rank of A.

Quasi-Unipotent monodromy/$\mathbb{C}$

Let $f: X \longrightarrow S$ be a smooth proper morphism of schemes, l.f.t/$\mathbb{C}$.
Fact : $X^{an} \xrightarrow{f^{an}} S^{an}$ is a fibre bundle.

Cor. $R^i f^{an}_* (\underline{\mathbb{Z}})$ are locally constant w/ finite stalk on S^{an}.
In particular, $\exists$ monodromy repn $\rho^i : \pi_1(S^{an}, s) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(H^i(X_s, \mathbb{Z}))$.

Thm. ρ^i sends "loops around ∞" to quasi-unipotent operators.
pf. Hodge theory, see later.

Defn. an element $g \in GL_n(\text{Field})$ is quasi-unipotent if g^k is unipotent.

"loops around ∞" : ① If S is a smooth curve, choose a smooth projective compactification $S \subseteq \bar{S}$. Then $\forall x \in \bar{S} \setminus S$, $\exists$ well defined conj. class in $\pi_1(S^{an}, s)$ consisting of loops around x.
② If $\dim S$ arbitrary, consider finite morphism $C \rightarrow S$, C smooth look at images of loops around infinity.

Étale cohomology version
$f: X \longrightarrow S$ smooth proper morphism of Noeth. schemes. Let ℓ prime invertible on S. Then we get $R^i f_{*, \acute{e}t}(\mathbb{Z}_\ell)$ locally constant and $R^i f_{\acute{e}t, *}(\mathbb{Z}_\ell)_{\bar{s}} = H^i_{\acute{e}t}(X_{\bar{s}}, \mathbb{Z}_\ell)$ f.g. $\mathbb{Z}_\ell$-mod.

$$\rho^i_\ell : \pi_1(S, \bar{s}) \longrightarrow \mathrm{Aut}_{\mathbb{Z}_\ell}(H^i_{\acute{e}t}(X_{\bar{s}}, \mathbb{Z}_\ell)).$$

Thm (Grothendieck) ρ^i_ℓ have quasi-unipotent local monodromy.
i.e. for every $\mathrm{Spec}(K) \longrightarrow S$ where $K = \text{f.f.}(A)$, where A is a dvr (may assume complete dvr) the action (provided ℓ invertible in A/m_A).

$$\rho^i_{X_K/K, \ell} : \mathrm{Gal}(K^{sep}/K) \longrightarrow \mathrm{Aut}_{\mathbb{Z}_\ell}(H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Z}_\ell)).$$

when restricted to an inertia subgp has the following properties:
ⓐ the image of wild inertia is finite.
ⓑ $I_t = I/P \simeq \prod_{\ell' \neq char(K_A)} \mathbb{Z}_{\ell'}$ (tame inertia) if $\tau \in I$ is an element mapping to a top. generator of I_t then $\rho^i_{X_K/K, \ell}(\tau)$ is q-unipotent.

Rmk1 it's a thm just about $\{1\} \subseteq P \subseteq I \subseteq D \subseteq \mathrm{Gal}(K^{sep}/K)$.
Rmk2 one can deduce thm before from this thm by using comparison thms.
Rmk3 immediate reduction to $S = \mathrm{Spec}(K)$, $K = $ f.f.A.

Lemma no nontrivial continuous gp hom between pro-p and pro-ℓ gps.

Lemma M a f.g. $\mathbb{Z}_\ell$ module. Endow M w/ ℓ-adic topology. Then $\mathrm{Aut}_{\mathbb{Z}_\ell}(M) = \mathrm{Aut}_{cont.}(M) = \varprojlim \mathrm{Aut}(M/\ell^n M)$ is a profinite gp, and the kernel $\mathrm{Aut}_{\mathbb{Z}_\ell}(M) \longrightarrow \mathrm{Aut}(M/\ell M)$ is a pro-ℓ gp.

Lemma. If X/K variety, then $\rho^i_{X/K, \ell}$ is continuous.
pf. by defn $H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Z}_\ell) = \varprojlim_n H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Z}/\ell^n\mathbb{Z})$. This is easy ~~by~~ for X/K curve, and in general we use dévissage!

Set up. A dvr w/ f.f. K, residue field k.

$X/\mathrm{Spec}(K)$ smooth, proper, ℓ prime #

$$\begin{tikzcd}
\{1\} \arrow[r, phantom, "\subseteq"] & P \arrow[r, phantom, "\subseteq"] \arrow[d] & I \arrow[r, phantom, "\subseteq"] \arrow[d] & D \arrow[r, phantom, "\subseteq"] & \mathrm{Gal}(K^{sep}/K) \arrow[d, "\rho^i_{X/K,\ell}"] & GL_{b_i}(\mathbb{Q}_\ell) \arrow[d, phantom, sloped, "\cong"] \\
& \rho^i(P) \arrow[r, phantom, "\leq"] & \rho^i(I) & & \mathrm{Aut}(H^i_{et}(X_{\bar{K}}, \mathbb{Z}_\ell)) \arrow[r] & \mathrm{Aut}_{\mathbb{Q}_\ell}(H^i_{et} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)
\end{tikzcd}$$

Thm If $\tau \in I$ maps to a top. gen. of $I_t = I/P$, then $\rho(\tau)$ is q-u.

pf. in case K has a finite # of ℓ-power roots of 1.

Lemma Assume K has finitely many ℓ-power roots of 1. Then $\exists\ \sigma \in D$, $\tau \in I$, s.t. τ maps to a top. gen. of I_t, $\sigma\tau\sigma^{-1}$ and τ^α map to the same element of I_t, where $\alpha \in \hat{\mathbb{Z}}^*$ is $\alpha = (\alpha_2, \cdots, \alpha_\ell, \cdots)$ w/ $\alpha_\ell \equiv 1 \pmod{\ell}$ yet $\alpha_\ell \neq 1$.

pf. $D \longrightarrow \mathrm{Gal}(K^{sep}/k)$

$$\begin{tikzcd}
I \arrow[r, "\theta_{can}"] \arrow[d] & \varprojlim_{p \nmid n} \mu_n(K^{sep}) \\
I_t \arrow[ur, "\sim"'] &
\end{tikzcd}$$

$\left(\cong \text{ noncan'ly } \prod_{\ell' \neq p} \mathbb{Z}_{\ell'}\right)$.

$\theta_{can}(\sigma\tau\sigma^{-1}) = \sigma(\theta_{can}(\tau))$.

just pick some "good" σ.

Cor. thm holds if K has only a finite # of ℓ-power roots of 1 and in fact it holds for any continuous

$$\begin{tikzcd}
\mathrm{Gal}(K^{sep}/K) \arrow[r, "\rho"] \arrow[dr] & \mathrm{Aut}_{\mathbb{Z}_\ell}(M) \\
& \mathrm{Aut}_{\mathbb{Q}_\ell}(V)
\end{tikzcd}$$

where M, V finite.

pf. look at $\rho(\tau) \sim \rho(\sigma\tau\sigma^{-1})$, pick σ, τ in Lemma.

$$\left.\begin{array}{l} \# \rho(P) < \infty \\ \sigma\tau\sigma^{-1}, \tau^\alpha \text{ same in } I/P \end{array}\right\} \Rightarrow \rho((\sigma\tau\sigma^{-1})^N) = \rho((\tau^\alpha)^N). \text{ for some } N \geq 0$$

So we then look at eigenvalues of $(\rho(\tau))^N$. $\lambda_1, \cdots, \lambda_t \in \bar{\mathbb{Z}}_\ell^*$.

Then $\forall\ i;\ \exists\ j$, s.t. $\lambda_i^\alpha = \lambda_j$.

so $\forall i$, $\exists\ 0 \leq r \leq s$ s.t. $\lambda_i^{\alpha^r} = (\lambda_i^{\alpha^r})^{\alpha^s}$.

$\Rightarrow (\lambda_i^{\alpha^r})^{\alpha^s - 1} = 1$, $\alpha^r \neq 0$ and $\alpha^s - 1 \neq 0$.

$\lambda \in \bar{\mathbb{Z}}_\ell^*$ and $\lambda^\beta = 1$ w/ $\beta \neq 0$, then λ is a root of 1.
↑ by looking at ℓ-component.

Example: K local field, $A = \mathcal{O}_K$, $X \longrightarrow \mathrm{Spec}(K)$ abelian vty of dim g. Tate module $T_\ell X = \varprojlim X[\ell^n](\bar{K}) \underset{\text{cont.}}{\supset} \mathrm{Gal}(K^{sep}/K)$.
$\cong \mathbb{Z}_\ell^{\oplus 2g}$.

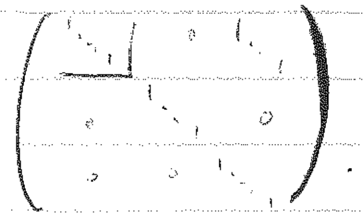
Then we have shown, after replacing K by some finite sep. extn, $\rho(P) = \{1\}$, ρ of inertia is gen. by a unipotent operator.

Additional fact: each Jordan block is 1×1 or 2×2.

Rmk If X has good reduction mod (π), then $\rho(I) = \{1\}$.

(bcs $T_\ell X \xrightarrow{\sim}$ ~~$T_\ell X/(\pi)$~~ $T_\ell\, X \otimes_{\mathcal{O}_K} \mathcal{O}_K/(\pi)$.

Picture $\rho(\tau) =$



Proof of Quasi-Unipotent Monodromy Thm, (equal char.)

Step 1 to prove Q-U.M.T, suffices to prove it when A is a cdvr, w/ $\kappa_A = \overline{\kappa_A}$.

reason: any A can be completed and you can always find an extn of dvr's $A \subseteq A'$ w/ $\kappa_{A'}$ alg. closed.

Step 2 reduce to action on $H^i_{ét}(X_{\bar K}, \mathbb{Z}/\ell\mathbb{Z})$ trivial.

Because we can change K to some L/K, s.t. $G(K^{sep}/L)$ acts trivially on $H^i_{ét}(X_{\bar K}, \mathbb{Z}/\ell\mathbb{Z})$ as it's finite and action is continuous.

Step 3 ~~Néron desingularization~~ Write $(A, \pi) = \mathrm{colim}(A_i, t)$ w/ A_i henselian, reg. local, κ_A f.g./prime field, t reg. parameter.

Néron desingularization (Tag 0BJ7) Let $R \in \Omega$ be an extn of dvr's w/ $e=1$, f.f.(Ω)/f.f.(R) separable, $\kappa(\Omega)/\kappa(R)$ separable. Then $\Omega = \mathrm{colim} A_i$ is a filtered colimit of smooth R-alg.

This is proved using Néron blow ups, essentially you have to be able to resolve singularities which is hard..

(so eq. char.)

Application Suppose A is a cdvr, $\kappa = \bar\kappa$ and A contains a field. Then (Cohen Str. Thm) $A \simeq \kappa[[t]]$ w/ $\kappa = \bar\kappa$. Then we apply Néron desingularization to $\mathbb{F}_p[[t]] \subseteq \kappa[[t]]$ or $\mathbb{Q}[[t]] \subseteq \kappa[[t]]$.

Cor. $A = \mathrm{Colim}\, A_i$, filtered colim, w/ A_i smooth $/\mathbb{F}_p[[t]]$ or $\mathbb{Q}[[t]]$

and t maps to uniformizer π of A.

Rmk. In particular, f.f.$(A) = K = A[\frac{1}{\pi}] = \mathrm{colim}\, A_i[\frac{1}{t}]$.

We can replace A_i by A_i^h (henselian of $(A_i)_{A_i \cap m_A}$), then we see that $(A, \pi) = \mathrm{colim}(A_i, t)$ (filtered), s.t.

- A_i henselian reg. local ring
- A_i/m_{A_i} is a f.g. field extn of $\mathbb{F}_p$ or $\mathbb{Q}$
- $t \in m_{A_i}$, $t \notin m_{A_i}^2$ (by smoothness)

Step 4. Get $X_i \longrightarrow \mathrm{Spec}(A_i[\frac{1}{t}])$ proper smooth, whose base change is X.

get: $\pi_1(\mathrm{Spec}(A_i[\frac{1}{t}])) \longrightarrow \mathrm{Aut}(H^k_{ét}(X_{\bar K}, \mathbb{Z}_\ell))$

Question: What's the structure of $\pi_1(\mathrm{Spec}(A_i[\frac{1}{t}])) = \pi_1(\mathrm{Spec}(A_i) \setminus V(t))$?

Generalized Abhyankar's Lemma: Let (A, m, κ) be a reg. henselian local ring. Let $t_1, \cdots, t_d \in m$ be a reg. system of parameters. Let $0 \le r \le d$. Then $\exists$:

$$\pi_1(\mathrm{Spec}(A[\tfrac{1}{t_1 \cdots t_r}])) \longrightarrow \pi_1^t \text{ w/ following properties.}$$

ⓐ kernel of this quotient is top. generated by pro-p-gp, $p = \mathrm{char}(\kappa)$, trivial if $\mathrm{char}(\kappa) = 0$.

ⓑ $\exists$ ses: $0 \longrightarrow \prod_{i=1}^{r} (\lim_{p \nmid n} \mu_n(K^{sep})) \longrightarrow \pi_1^t \longrightarrow \mathrm{Gal}(K^{sep}/K) \longrightarrow 1$

the action of $\mathrm{Gal}(K^{sep}/K)$ on $\prod_{i=1}^{r} (\lim_{p \nmid n} \mu_n(K^{sep}))$ is natural...

Rmk. π_1^t = <u>tame fund. gp.</u> of $(\mathrm{Spec}(A), D = V(t_1 \cdots t_r))$. There's a general definition of tame fund. gp for pairs (S, D) where $D \subseteq S$ is a n.c. divisor.

Idea of pf. map: $\pi_1(\mathrm{Spec}\,A[\frac{1}{t_1\cdots t_r}]) \longrightarrow \pi_1^t$ comes from considering Galois cover of $\bigcup_{p\nmid n}$ f.f. $(A[\sqrt[n]{t_1},\cdots,\sqrt[n]{t_r}])$ + ramification thy + purity + usual Abhyankar's Lemma.

<u>Step 5</u> We can increase i, so we may assume

$$\rho^k_{X_i/A_i,\ell} \text{ mod } \ell \text{ is trivial.}$$

so image is pro-ℓ.

<u>Step 6</u> In the situation above, we see it satisfies the condition in gen. Abhyankar's Lemma, where $r=1$. So then

$$\pi_1(\mathrm{Spec}(A[\tfrac{1}{t_1}]) \xrightarrow{\rho} GL_n(\mathbb{Q}_\ell) \text{ factors thru } \pi_1^t,$$

and $\ker(\pi_1^t \longrightarrow \mathrm{Gal}(\kappa_i^{sep}/\kappa_i))$ is mapped to q-unipotent. (last part follows by the argument before as κ_i doesn't contain all ℓ-power roots of unity).

$$\pi_1(K^{sep}/K) = \pi_1(\mathrm{Spec}\,A[\tfrac{1}{\pi}]).$$

$$I^t = \pi_1^t(\mathrm{Spec}(A), V(\pi)) = \mathrm{colim}\,\ker(\pi_1^t \longrightarrow \mathrm{Gal}(\kappa_i^{sep}/\kappa_i))\ldots$$

Left over: mixed char. cdvr A, $\kappa=\bar\kappa$.

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r, phantom, "\overset{e}{\subseteq}"] & A \ni \pi \\ & W(\kappa) \arrow[u, phantom, "\cup\, e"] \end{tikzcd}$$

π satisfying Eisenstein eq. over $W(\kappa)$ $\pi^e + \lambda_1\pi^{e-1} + \cdots + \lambda_e = 0$.

Idea: construct

$$\begin{tikzcd} R \arrow[r] & A \\ W(\kappa_0) \arrow[r] \arrow[u, phantom, "\cup"] & W(\kappa) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where
- $\kappa_0 \subseteq \kappa$ perfection of f.g. field extn of $\mathbb{F}_p$.
- $R \simeq W(\kappa_0)[x]/(x^e + \cdots + \lambda'_e)$ w/ λ'_i close to λ_i.

Zink, Cartier theorie, about Witt ring

Proof of the fact in mixed char: $(A,\pi) = \mathrm{colim}(A_i, t)$ (filtered)

κ_i is a purely insep. extn of a f.g. field / prime field

Step 1. $\forall$ every κ perfect, char $p>0$, $\exists$ ~~char~~ canonical cdvr $W(\kappa)$ w/ uniformizer p, s.t. :

$\forall$ any complete local ring (B, m_B) and $\kappa \longrightarrow B/m_B$, $\exists!$ lift $W(\kappa) \longrightarrow B$.

Step 2. we get $W(\kappa) \longrightarrow A$. Then A will be a flat $W(\kappa)$ alg. w/ ramification index e.

$$\pi^e + \lambda_1\pi^{e-1} + \cdots + \lambda_e = 0 \quad \lambda_i \in W(\kappa).$$

Krasner's Lemma. $\exists N>0$, s.t. if $\lambda'_1, \cdots, \lambda'_e \in W(\kappa)$ s.t. $\lambda_i - \lambda'_i \in p^N W(\kappa)$ then $P(X) = X^e + \lambda'_1 X^{e-1} + \cdots + \lambda'_e$ has a root $\pi' \in A$, w/ $\pi' \equiv \pi \text{ mod } m_A^2$.

Step 3. Given $\lambda \in W(\kappa)$ and $N>0$ we can find $\kappa_0 \subseteq \kappa$, κ_0 is perfection of f.g. extn of $\mathbb{F}_p$. and $\lambda' \in W(\kappa_0)$, s.t. $\lambda - \lambda' \in p^N W(\kappa)$, where $W(\kappa_0) \longrightarrow W(\kappa)$ comes from universal property of $W(-)$ and $\kappa_0 \hookrightarrow \kappa$.

For $N=1$, we just choose $\mathbb{F}_p(t) \hookrightarrow \kappa$, $t \longmapsto \bar\lambda$, κ_0 to be the perfection of $\mathbb{F}_p(t)$, λ' preimage of t...

Then ~~$\frac{\lambda-\lambda'}{p}$~~ apply to $\frac{\lambda-\lambda'}{p}$, we solve $N=2$, so on so forth.

~~Step 4. We get $W(\kappa_0)[x]/$~~

Step 4 Pick N as in Krasner, pick $\kappa_0, \lambda_1', \dots, \lambda_e'$ by Step 3.
Set $A_0 = W(\kappa_0)[x]/(x^e + \lambda_1' x^{e-1} + \dots + \lambda_e')$

Because we started with an Eisenstein polynomial, we see that A_0 is a cdvr with uniformizer $\bar{x}$.

We get
$$\begin{tikzcd} A_0 \arrow[r, dashed, "\text{Krasner}"] & A \\ W(\kappa_0) \arrow[u, phantom, "\cup"] \arrow[r] & W(\kappa) \arrow[u, phantom, "\cup"] \end{tikzcd} \qquad \bar{x} \longmapsto \pi'.$$

Step 5. Néron desingularization applies to $A_0 \longrightarrow A$.
Thus $(A, \pi') = \operatorname{colim}(A_i, t)$.

Birat'l invariance of π_1:

Lemma $f: X \longrightarrow Y$ birat'l proper morphism of varieties w/ X normal and Y nonsingular. Then $\pi_1(X) \simeq \pi_1(Y)$.

pf. $U \subseteq X$, largest open, ~~then~~ s.t. $f|_U : U \simeq f(U)$. Then $\operatorname{codim}(Y \setminus f(U), Y) \geq 2$.

By purity
$$\begin{tikzcd} \pi_1(Y) & \pi_1(f(U)) \arrow[l, "\simeq"'] \\ \pi_1(X) \arrow[u] & \pi_1(U) \arrow[l, two heads] \arrow[u, equal] \end{tikzcd}$$
uses that X is normal, so both sides are pro-quotient of $G(K(X))$.

Cor. π_1 is only about function field if we have resolution of singularity (if we can find smooth model).

Cor. If X is rat'l smooth proj. vty$/k = \bar{k}$, then $\pi_1(X) = \{1\}$

Recall (not discussed). $X \longrightarrow Y$ flat + proper morphism of vties all of geom. fibres are connected + reduced.
Then $\exists$ exact sequence:
$$\pi_1(X_{\bar{t}}) \longrightarrow \pi_1(X) \longrightarrow \pi_1(Y) \longrightarrow 1.$$

Special case $Y = \mathbb{P}^1_k$, X sm. proj. surface $/k = \bar{k} \Longrightarrow \pi_1(X)$ is a quotient of $\pi_1(X_{\bar{t}})$ $\forall t \in \mathbb{P}^1$.

E.g. If one fibre is a tree of $\mathbb{P}^1$'s, then $\pi_1(X) = \{1\}$.

e.g. $g(X_{\bar{\eta}}) = 0$.

E.g. $g(X_{\bar{\eta}}) = 1$ and fibres are at worst nodal $\xrightarrow[\text{of subtg}]{\text{grain}}$
either $\begin{pmatrix} X = E \times \mathbb{P}^1 \\ \text{and } \pi_1(X) = \pi_1(E) \end{pmatrix}$ or $\pi_1(X)$ finite cyclic.

no sing. fibre $\Leftrightarrow$ $X = E \times \mathbb{P}^1 \Leftrightarrow$ j-inv. constant.

rest case $\Longrightarrow \exists$ bad fibre $\Rightarrow \exists \bar{t}$, s.t. $\pi_1(X_{\bar{t}}) = \hat{\mathbb{Z}} \Rightarrow \pi_1(X)$ pro-cyclic.

conn. étale $\mathbb{Z}/n$ covering $\longleftrightarrow$ $\mathcal{L}$ invertible on X w/ $\mathcal{L}^{\otimes n} \simeq \mathcal{O}_X$ but $\mathcal{L}^{\otimes m} \neq \mathcal{O}_X$ for all $0 < m < n$.

$$\begin{matrix} X' \\ \downarrow g \\ X \end{matrix} \quad \mathbb{Z}/n \text{ étale} \Rightarrow \mathbb{Z}/n\mathbb{Z} \circlearrowright g_*(\mathcal{O}_{X'}) \simeq \bigoplus_{\chi \in \mathbb{Z}/n\mathbb{Z} \to k^\times} \mathcal{L}_\chi.$$

Mordell-Weil-Lang-Néron $\underline{\operatorname{Pic}}^{\circ}_{X_\eta/\eta}(K(\eta))$ f.g. abelian gp.

(Arithmetic Geometry, expository of Artin) about Lipman's Thm

Defn. Let Y be a Noetherian integral scheme. A resolution of singularity is a proper birational morphism of integral schemes $f: X \longrightarrow Y$ such that X is regular.

Thm. (Lipman) Let Y be an integral Noetherian 2 dim'l scheme, such that ① the normalization morphism $Y^{\nu} \longrightarrow Y$ is finite ② Y^{ν} has finitely many singular points $y_1, \cdots, y_n$ and $\widehat{\mathcal{O}}_{Y^{\nu}, y_i}$ is normal.
Then $\exists$ resolution of singularities of Y (conversely also)

Rmk If Y is of finite type over field or $\mathbb{Z}$ or char 0 Dedekind domain or a complete Noetherian local ring, then ① & ② hold. (quasi-excellent schemes).

~~Ari~~

Notation (fix) R dvr w/ $K = \text{f.f.}(R)$, C/K proper, smooth, geometrically connected.

Prop. There exists
$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ K \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$
w/ f flat proj. and X regular.

proof. Choose $C \hookrightarrow \mathbb{P}^n_K$ and let $Y \subseteq \mathbb{P}^n_R$ be the Zariski closure. Fact: Y satisfies condition in Lipman's Thm.
we get a X res. of. sing. of Y.

Since $C \subseteq Y$ open and C reg. of dim 1. We see that $X \longrightarrow Y$ is isom. over C. Therefore $X_K \cong C$.

E.g. $\mathbb{Z}_p \longrightarrow \mathbb{Z}_p[x,y]/(xy(x+y)^p-p)$.

Embedded resolution:
Let Y be a 2-dim'l scheme. Let $Z \subseteq Y$ be a closed subscheme, all of whose irreducible components Z_i are either pts or 1-dim'l scheme whose normalization is finite.
Then $\exists$ sequence of blow ups $X \xrightarrow{f} Y$, s.t.
$f^{-1}(Z)_{red}$ is a s.n.c.d.

Prop. There exists

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

s.t. f is proj. + flat, X regular, $(X_s)_{red}$ is s.n.c.d.

Notation: $X \xrightarrow{f} \mathrm{Spec}(R) = S$ as $\quad C_1, \cdots, C_n$ irred. comp. of X_s.
$X_s = \sum r_i C_i$ as Cartier divisors
X_s reduced $\Leftrightarrow$ $r_i = 1$ for all $1 \le i \le n$.

<u>Claim</u> If $\mathrm{char}(\kappa(s)) = 0$, then setting $r = \mathrm{lcm.}(r_i)$, $R' = R[\pi']$ w/ $\pi' = \pi^{1/r}$.
$X' =$ normalization of base change $X \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$.
Then $X' \longrightarrow \mathrm{Spec}(R')$ is a semi-stable curve over R'.

Lemma. (Addendum) If X is a semi-stable curve over a dvr R w/ smooth geometric fiber, then
ⓐ singularities of X as a surface are (A_k), $k \geq 1$
ⓑ There exists a repeated blow up $X_n \to X_{n-1} \to \cdots \to X_0 = X$ s.t. X_n is a semi-stable S-variety.

Rmk Claim is also true if
- $\mathrm{Char}(\kappa(s)) = p > 0$
- $p \nmid r_i$
- $C_i \longrightarrow \mathrm{Spec}(\kappa(s))$ smooth
- $x \in C_i \cap C_j$ $(i \neq j)$ then $\kappa(x)/\kappa(s)$ separable.

Idea: describe étale local structure of X + use that normalization commutes with étale localization.

Let $x \in C_i \cap C_j$ $(i \neq j)$ with
① $\kappa(x)/\kappa(s)$ sep.
② r_i, r_j prime to $\mathrm{char}(\kappa(s))$.

$R \longrightarrow \mathcal{O}_{X,x} \supseteq m_x$
Let $a, b \in m_x$ be local equations for C_i, C_j, $\overset{n}{\cancel{m}} = r_i$, $m = r_j$.
Then $\pi = u \cdot a^n \cdot b^m \quad u \in \mathcal{O}_{X,x}^*$.

Because $\mathrm{char}(\kappa(s)) = \mathrm{char}(\kappa(x)) \nmid n \Rightarrow$ after replacing X by an étale cover we may assume $\pi = a^n b^m$.

$$\begin{tikzcd}
\kappa(x)/\kappa(s) \text{ sep.} \Rightarrow \mathcal{O}_{X,x} & R[u,v]/(u^n v^m - \pi) \arrow[l] \\
a & u \arrow[l, mapsto] \\
b & v \arrow[l, mapsto]
\end{tikzcd}$$

defines an étale morphism.

Final step. compute étale local structure of

$$A'_{n,m,d} = \operatorname{Spec}\left(R'[u,v]/(u^n v^m - (\pi')^d)\right)^{\text{norm}}.$$

<u>Step 1</u> if $e = \gcd(n,m) > 1$ and $\zeta_e \in R'$.

then $A'_{n,m,d} = \prod_{i=1}^{e} A'_{n/e,\, m/e,\, d/e}$

<u>Step 2</u> $\gcd(n,m) = 1$ $\quad u^n = \left(\frac{\pi'^{d/m}}{v}\right)^m \quad v^m = \left(\frac{\pi'^{d/n}}{u}\right)^n$.

So $\exists\ s,t \in A'_{n,m,d}$, s.t.

$t^m = u,\ t^n = \frac{\pi'^{d/m}}{v},\ s^n = v,\ s^m = \frac{(\pi')^{d/n}}{u}$.

So $A'_{n,m,d} = R'[s,t]/(st - (\pi')^{d/mn})$.

Semi-stable reduction for curves.

Defn. Let X be a locally Noetherian scheme. A s.n.c.d. is an eff. Cartier divisor $D \subseteq X$ s.t. $\forall p \in D$, the local ring $\mathcal{O}_{X,p}$ is reg., and $\exists$ a reg. system of parameters $x_1, \dots, x_d \in \mathfrak{m}_p$ and $1 \le r \le d$, s.t. $D \times_X \operatorname{Spec}(\mathcal{O}_{X,p})$ is $V(x_1 \cdots x_r)$.

lemma $\iff$
- $D \subseteq X$ eff. Cartier
- each irred. component D_i of D is regular
- $D_{i_1} \cap \dots \cap D_{i_t}$ is reg. of codim'n t in X.
- $D = \sum_{i \in I} D_i$ (i.e., D is reduced).

Defn. X locally Noeth., then $D \subseteq X$ eff. Cartier divisor is a n.c.d. if $\exists$ $\{U_k \to X\}_{k \in K}$ étale covering, s.t., $D \times_X U_k$ is a s.n.c.d. in U_k $\forall k \in K$.

Defn. R dvr. $S = \operatorname{Spec}(R)$ trait $\ni \eta, s$.

(a) An S-vty is an integral scheme X, separated and of finite type over S, with $X_\eta \neq \phi$.

(b) $X_s = f^{-1}(s) = X \otimes \kappa(s) = V(\pi) \subseteq X$.

(c) we say X is strictly semi-stable over S if

(C_1) X_η smooth over $\kappa(\eta)$.

(C_2) X_s reduced

(C_3) each irred. comp. X_i of X_s is an effective Cartier on $X \xRightarrow{(C_2)} X_s = \sum X_i$.

(C_4) for all nonempty $J \subseteq I$ we have $X_J = \bigcap_J X_j$ is smooth over $\kappa(s)$ and codim $= \# J$ in X.

(in particular, X_s is a s.n.c.d on X).

Fact If $x \in X_s$, then $\widehat{\mathcal{O}}_{X,x} \cong B[\![t_1, \cdots t_r]\!]/(t_1 \cdots t_r - \pi)$. where B is a complete local R-alg. which is formally smooth over R.

Fact If $\kappa(s)$ is perfect, then $(C_2), (C_3), (C_4) \iff X_s$ is s.n.c.d in X

Defn. We say X is semi-stable over S if étale locally on X we have X is s. s-s over S.

Implication

$$\begin{tikzcd}
(X_s)_{red} \text{ n.c.d.} & X_s \text{ n.c.d} \arrow[l, Rightarrow] & X \text{ s.s.}/S \arrow[l, Rightarrow] \\
(X_s)_{red} \text{ s.n.c.d} \arrow[u, Rightarrow] & X_s \text{ s.n.c.d} \arrow[l, Rightarrow] \arrow[u, Rightarrow] & X \text{ s s-s.}/S \arrow[l, Rightarrow] \arrow[u, Rightarrow]
\end{tikzcd}$$

Defn. S scheme, A semi-stable curve over S is $f: X \to S$ flat + proper + f.p. s.t. all geom. fibres are connected, dim 1 w/ singularities at worst nodes.

Note: $X/k = \bar{k}$, f.t., dim $= 1$, $x \in X$ closed, is a node, if $\widehat{\mathcal{O}}_{X,x} \cong k[\![u, v]\!]/(uv)$.

Defn. Split semi-stable curve X over a field k means
- it's a semi-stable curve over k
- all irred. comp. are geom. irred.
- nodes are all k-rat'l.

Split semi-stable curve$/S \iff$ semi-stable curve$/S$ s.t. all fibres split.

Lemma. If R is a dvr, $S = \mathrm{Spec}(R)$, $X \longrightarrow S$ proper S vty of rel dim 1 w/ geom. connected fibres

X/S is semi-stable as an S-vty $\Rightarrow$ X is a semi-stable curve $/S$.

Rmk for the converse it's true that after blowing up, a s-s curve (+ all the conditions) will be s-s S-vty.

Given R dvr w/ fraction field K. C/K smooth proper, geom. connected curve, there $\exists$ $R \subseteq R'$ extn of dvrs

$X' \longrightarrow S' = \mathrm{Spec}(R')$ a s. s-s. S'-vty.

$C \otimes_K K' \cong X' \otimes_{R'} K'$.

Additional properties we'd like

- K'/K finite sep.
- $\exists Y \longrightarrow \mathrm{Spec}(B)$ where $B =$ integral closure of R in K', s.t. $Y \otimes_B B_{m'}$ is s. s-s vty over $B_{m'}$, $\forall$ $m' \subseteq B$ maximal.

(a standard argument allows one to work with one m' at a time).

Next Lecture was taken mistakingly on the next of Wei's Lecture...

$$J = \mathrm{Jac}(C) \quad T_\ell(J) = \varprojlim J[\ell^n](K^{sep}) \overset{\rho}{\circlearrowleft} I \supseteq P$$

If C has semi-stable reduction, then ρ is unipotent but base change by $\sqrt[d]{\pi}$ can't kill the wild inertia.

Method of Artin - Winters

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(K) \arrow[r, hook] & \mathrm{Spec}(R) \end{tikzcd}$$

where
- R, dvr w/ f.f. K.
- C smooth, proj., geometrically connected $/K$
- f proper + flat
- X regular

Let $C_1, \cdots, C_n$ be the irreducible components of X_s. Then $X_s = \Sigma r_i C_i$ as Cartier divisors. Let $r = \gcd(r_1, \cdots, r_n)$ (can be >1).

Lemma X_s is geom. connected $/\kappa(s)$.

'cause $X \to$ ~~R~~S is ~~[illegible]~~ proper

pf. $H^0(C, \mathcal{O}_C) = K \Rightarrow f_* \mathcal{O}_X \overset{\cdot}{=} \mathcal{O}_S \Rightarrow X_s$ geom. connected.

For $\mathcal{L} \in \mathrm{Pic}(X)$ and $i \in \{1, \cdots, n\}$ we define

$$\mathcal{L} \cdot C_i = \deg(\mathcal{L}|_{C_i}).$$

If $D \subseteq X$ is an (eff) Cartier divisor, define

$$D . C_i = \mathcal{O}_X(D) \cdot C_i$$

Lemma $(\Sigma r_i C_i) \cdot C_j = 0$ for all j.

Let $\Lambda = \oplus \mathbb{Z} C_i$, $F = \Sigma r_i C_i$. $\Lambda \times \Lambda \xrightarrow{\alpha} \mathbb{Z}$ symmetric bilinear form.

Lemma The pairing is semi-negative definite and if $Z \in \Lambda$ w/ $Z^2 = 0$,
then $Z \in \mathbb{Z}(\frac{1}{r}F)$.

pf. say $Z = \Sigma s_i C_i$, then $Z^2 = \sum_i s_i C_i \left(\sum_j s_j C_j\right) =$

$$= \sum_i s_i C_i \left(\sum_j s_j C_j - \frac{s_i}{r_i} \sum_j r_j C_j \right)$$

$$= \sum_i s_i C_i \left(\sum_{j \neq i} \left(s_j - \frac{s_i r_j}{r_i} \right) C_j \right)$$

$$= \sum_{i \neq j} \frac{s_i}{r_i} (r_i s_j - r_j s_i) C_i \cdot C_j .$$

$$= \sum_{i < j} - \frac{(r_i s_j - r_j s_i)^2}{r_i r_j} C_i \cdot C_j .$$

Now use X_s connected.

There is an exact sequence

$$0 \to \mathbb{Z} \to \Lambda \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(C) \longrightarrow 0$$

$$1 \longmapsto F$$

Lemma. The restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_s)$, if n prime to $\chi(s)$, then
$\mathrm{Pic}(X)[n] \longrightarrow \mathrm{Pic}(X_s)[n]$ is injective.

Set $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z})$, then get an exact sequence

$$0 \to \mathbb{Z} \to \Lambda \longrightarrow \Lambda^* \longrightarrow G \longrightarrow 0.$$

$$1 \longmapsto \tfrac{1}{r} F$$

$$D \longmapsto \alpha(D, -)$$

$G = \mathbb{Z} \oplus G_{tors}$, and $\# G_{tors} < \infty$

Lemma There is an exact sequence

$$0 \to \mathbb{Z}/\gcd(n,r)\mathbb{Z} \longrightarrow \mathrm{Pic}(X)[n] \longrightarrow \mathrm{Pic}(C)[n] \longrightarrow \frac{\alpha^{-1}(n\Lambda^*)}{n\Lambda + \mathbb{Z}F}$$

pf. $\ker\left((\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(C) \right) = \Lambda/\mathbb{Z}F$.

The torsion is $\mathbb{Z}(\frac{1}{r}F)/\mathbb{Z}F = \mathbb{Z}/r\mathbb{Z}$

So $(\mathbb{Z}/r\mathbb{Z})[n] \cong \mathbb{Z}/\gcd(n,r)\mathbb{Z}$

Suppose $\mathcal{L}_C \in \mathrm{Pic}(C)[n]$. Pick $\mathcal{L} \in \mathrm{Pic}(X)$ w/ $\mathcal{L}_C \cong \mathcal{L}|_C$.

Then $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(\Sigma a_i C_i)$ for some $\Sigma a_i C_i \in \Lambda$,
well-defined modulo $\mathbb{Z}F$.

Moreover $(\Sigma a_i C_i) \cdot C_j = n(\mathcal{L} \cdot C_j) \in n\mathbb{Z}$, hence $\Sigma a_i C_i \in \alpha^{-1}(n\Lambda^*)$,
well-defined modulo $\mathbb{Z}F$.

On the other hand, if we replace $\mathcal{L}$ by $\mathcal{L}(\Sigma b_i C_i)$,
then $(a_i) \longmapsto (a_i + n b_i)$.

So we get well-defined class $\frac{\alpha^{-1}(n\Lambda^*)}{n\Lambda + \mathbb{Z}F}$

Lemma There is an exact sequence

$$0 \to \frac{\alpha^{-1}(n\Lambda^*)}{n\Lambda^*} \longrightarrow \frac{\Lambda}{n\Lambda} \xrightarrow{\alpha} \frac{\Lambda^*}{n\Lambda^*} \longrightarrow \frac{G}{nG} \longrightarrow 0$$

and hence $\# \mathrm{Pic}(X)[n] \geq \frac{\# \mathrm{Pic}(C)[n]}{\#(G_{tors}/nG_{tors})}$.

pf. $\# \mathrm{Pic}(X)[n] \geq \frac{\mathrm{Pic}(C)[n] \cdot \gcd(n,r)}{\#\left(\frac{\alpha^{-1}(n\Lambda^*)}{n\Lambda + \mathbb{Z}F}\right)} = \frac{\mathrm{Pic}(C)[n] \cdot n}{\#\left(\frac{\alpha^{-1}(n\Lambda^*)}{n\Lambda}\right)}$

$$= \frac{\mathrm{Pic}(C)[n] \cdot n}{\#(G/nG)} = \frac{\mathrm{Pic}(C)[n]}{\#(G_{tors}/nG_{tors})}.$$

Example application: If R is strictly henselian and ℓ prime $\ell \in \kappa(s)^*$ and $\mathrm{Gal}(K^{sep}/K) = I$ acts trivialy on $J[\ell^m](K^{sep})$ $\forall m \geq 1$, then we conclude that

$$\# \mathrm{Pic}(X_s)[\ell^m] \geqslant (\ell^m)^{2g} / \text{fixed constant}.$$

we will see later this implies X_s is a tree of smooth curves $+ \Sigma g_i = g$.

Example Again, say, ℓ prime to $\mathrm{char}(\kappa(s))$ and $\mathrm{Pic}(C)[\ell] \cong \mathbb{F}_\ell^{2g}$. Then we get $\dim_{\mathbb{F}_\ell}(\mathrm{Pic}(X_s)[\ell]) \geq 2g - \dim \mathbb{F}_\ell (G_{tors}/\ell G_{tors})$.

1st part of Artin-Winters "bounding the graphs".

It suffices to assume residue field of the dvr is alg. closed.

Pick a model as before

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & X \arrow[d] & \geq X_s = \Sigma r_i C_i \arrow[d] \\ \mathrm{Spec}(K) \arrow[r, hook] & \mathrm{Spec}(R) & \ni s \end{tikzcd}$$

Then we let $\omega_X = \mathcal{O}_X(K_X) = \mathcal{O}_X(K)$ be the dualizing sheaf of X.

Defn. The type T of X consists of:

- n, # of irred comp. of X_s.
- $(m_{ij}) = C_i \cdot C_j$ symm. $n \times n$ matrix.
- $k_i = K_X \cdot C_i$
- r_i

Lemma. $P_a(C_i) = \dim_{\kappa(s)} H^1(C_i, \mathcal{O}_{C_i})$, then $P_a(C_i) = \frac{1}{2}(C_i^2 + K \cdot C_i) + 1$

pf. adjunction gives $\omega_{C_i} = \mathcal{O}_X(C_i + K)|_{C_i}$

$\deg \omega_{C_i} = -2\chi(C_i, \mathcal{O}_{C_i}) = -2(1 - P_a(C_i))$. □

Lemma. $g(C) = 1 + \frac{1}{2} \sum_i r_i k_i$

pf. $2(g(C)) - 2 = \deg \omega_C = \deg(\mathcal{O}_X(K)|_C) = \deg(\mathcal{O}_X(K)|_{X_s})$

$= \Sigma r_i K \cdot C_i = \Sigma r_i k_i$

Defn. An <u>abstract type</u> T is a collection

m_{ji}

$T = (n; m_{ij}; k_i; r_i)$ s.t. $n \geq 1$, $m_{ij} \geq 0$ if $i \neq j$,

$\Sigma r_i m_{ij} = 0$ $\forall j$. Γ associated to (m_{ij}) is connected

$p_a(C_i) \overset{\text{def}}{=\!=} \frac{1}{2}(m_{ii}+k_i)+1 \geq 0$ is an integer, $r_i \geq 1$.

Defn. The genus of an abstract type is
$$g = 1+\frac{1}{2}\sum r_i k_i$$

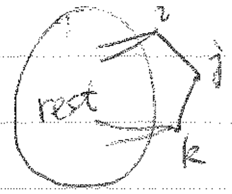
Lemma. (m_{ij}) is semi-neg. definite and $Z^2=0 \Leftrightarrow Z \in \mathbb{Z}(\frac{1}{r}\sum r_i C_i)$
where $r = \gcd(r_i)$

Observe: $m_{ii}=0 \Leftrightarrow r_i=1$. In this case $g = 1+\frac{1}{2}r_i k_i$
for some i
This bounds possibilities of T unless $g=1$.

Otherwise $m_{ii}<0$ $\forall i \Rightarrow k_i \geq -1$ $\forall i$.

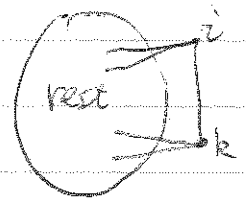
Special case:
- $m_{ii}=k_i=-1$, a (-1)-curve.
- $k_i=0$, $m_{ii}=-2$, a (-2)-curve.
- $k_i \geq 1$, $m_{ii}<0$.

Equivalence on types: generated by



$r_i=r_j=r_k$
$m_{ij}=m_{jk}=1$
$m_{ii}=m_{jj}=m_{kk}=-2$
$m_{ik}=0$
$k_i=k_j=k_k=0$

$\sim$



Thm Fix $g \geq 2$. $\exists$ finitely many equivalence classes of abstract types without (-1)-curves having genus g.

pf. Write $\sum r_i C_i = A+B$.
$I = \{i \mid k_i=0\}$, $J=\{i \mid k_i>0\}$
$A = \sum_{i\in I} r_i C_i$, $B = \sum_{j\in J} r_j C_j$.

Then $g = 1+\frac{1}{2}\sum_{i\in J} r_i k_i$
$$\Rightarrow \begin{cases} \#J \leq 2g \\ r_i k_i \leq 2g \quad \forall i\in J. \end{cases}$$
$\frac{1}{2}(m_{ii}+k_i)+1 \geq 0 \Rightarrow m_{ii} \geq -2g-2 \quad \forall j \in J$.
$(C_i+C_j)^2 \leq 0 \xRightarrow{i,j\in J} C_iC_j \leq 2g+2$, $\forall i,j\in J$.

Sublemma: ~~Sublem~~ There is a finite list of possibilities for B-part.

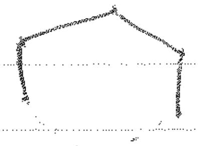
E.g. $B^2 \geq -(2g)^2\cdot 2g\cdot(2g+2)$.
Suppose $j\in J, i\in I$, then
$$0 = C_j(r_jC_j+\cdots) \geq r_jC_j^2+r_iC_iC_j$$
$$\Rightarrow r_iC_iC_j \leq -r_jC_j^2 \leq 2g(2g+2).$$

Conclusion: If C_i and C_j meet, $i\in I$, $j\in J$, then r_i and C_iC_j is bdd by $4g(g+1)$.
$$\left.\begin{array}{l} 0=A(A+B) \\ 0=B(A+B) \end{array}\right\} \Rightarrow A^2=-AB=B^2 \geq -(2g)^2\cdot \underset{\substack{\shortparallel \\ C(g)}}{2g\cdot(2g+2)}.$$
$$\Rightarrow A^2 \geq -C(g),\ AB \leq C(g).$$
So # connected comp. of A is bdd.

Rmk. As $g\geq 2$, $B\neq\phi$, so intersection pairing $|_A$ is neg. def.

E.g. $\left.\begin{array}{l} i,j\in I,\ i\neq j \\ (C_i+C_j)^2<0 \end{array}\right\} \Rightarrow -2+2C_iC_j-2<0 \Rightarrow C_iC_j = \begin{cases} 0 \\ 1 \end{cases}$.

Picture:



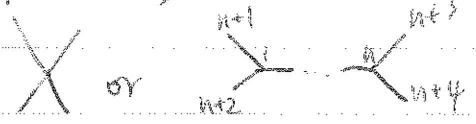
Lemma. ① A has no loop

② A has no vertice w/ valence ≥ 4

③ every connected component of A has at most 1 triple point.

pf.

$(C_1 + C_2 + \cdots + C_n)^2 = 0$.



$(C_1 + \cdots + C_n + \frac{1}{2}(C_{n+1} + \cdots + C_{n+4}))^2 = 0$.

w/ extra work we will see conn. comp. of A can only be

A_n, D_n, E_6, E_7, E_8.

Left to show: when n is "very large", r_i's are constant on 3 consecutive indices. Let's look at A_n type:

Let i be an index w/ r_i maximal.

① ~~$1 < i < n$ and~~ C_i does not meet B.

$$-2r_i + r_{i-1} + r_{i+1} = 0.$$

$$\Rightarrow r_{i-1} = r_i = r_{i+1}.$$

Defn Let G be a finite generated abelian group, let $c \geq 1$, then

$$\rho_c(G) = \min\{r \mid \exists H \leq G \text{ subgp of index dividing } c \text{ st. } H \text{ can be gen. by } r \text{ elements}\}$$

E.g. $\rho_1(G) =$ min # of generators of G.

Lemma. If $0 \to G_1 \to G_2 \to G_3 \to 0$ is exact, then

$$\rho_{cc'}(G_2) \leq \rho_c(G_1) + \rho_{c'}(G_3).$$

$$\rho_c(G_2) \geq \rho_c(G_3).$$

E.g. If $l \nmid c$ prime, then $\dim_{\mathbb{F}_l}(G/lG) \leq \rho_c(G)$.

pf. $H/lH \simeq G/lG$

Thm $\forall g \geq 0$, $\exists$ a $c = c(g)$, s.t. if T is an abstract type of genus g, then $\rho_c(G) \leq 1 + \beta$, where β is the 1st betti number of graph T, $G = \operatorname{coker}(\Lambda \xrightarrow{(m_{ij})} \Lambda^*)$, $\Lambda = \bigoplus_{i=1}^{n} \mathbb{Z} C_i$.

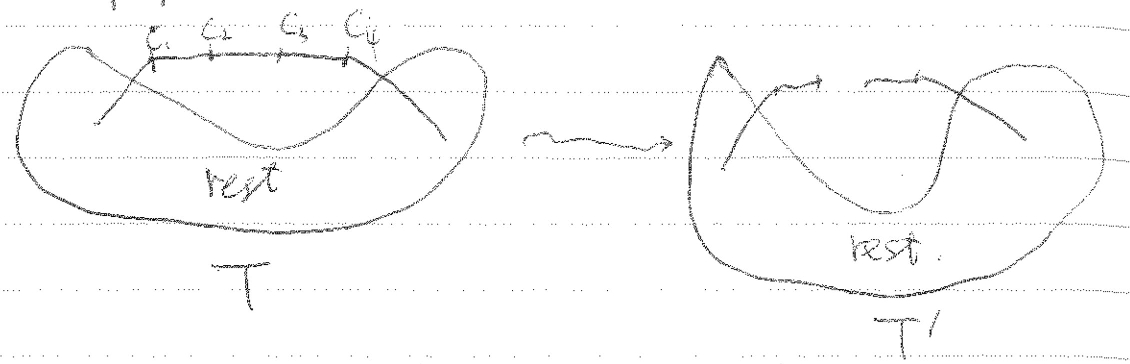
pf. Step 1. Argue that (-1) curves can be "contracted".

Step 2. do $g = 0, 1$ separately.

Step 3. use boundedness of last lecture to do $g \geq 2$ by induction on g, up to equivalence, only finitely many, so we can pick c.

If the graph contains a chain of (-2) curves with length ≥ 4, same multiplicity:

we perform cut:



Letting $X_1, \cdots, X_n$ be the basis of Λ^* dual to $c_1, \cdots, c_n$ in Λ.
we see

$$M = \mathrm{Im}\left(\Lambda \xrightarrow{(m_{ij})} \Lambda^*\right) = \mathrm{span}\begin{pmatrix} \sum_j m_{ij}X_j \quad i \neq 2,3 \ ; \\ X_1 - 2X_2 + X_3 \ ; \\ X_2 - 2X_3 + X_4 \end{pmatrix}.$$

$$M' = \mathrm{Span}\left(\sum m_{ij}X_j \ , \ i \neq 2,3 \ ; \ X_1 - X_2 \ ; \ -X_3 + X_4\right)$$

see $M' \subseteq M + \mathbb{Z}(X_2 - X_3)$

$$\begin{tikzcd} \Longrightarrow G = \Lambda^*/M \arrow[r] & \Lambda^*/M+\mathbb{Z}(X_2-X_3) = G/\langle X_2 - X_3 \rangle. \\ & G' = \Lambda^*/M'. \arrow[u] \end{tikzcd}$$

$$P_c(G) \leq P_c(G/\langle X_2 - X_3\rangle) + 1 \leq P_c(G') + 1 \underset{\substack{\text{If } T' \text{ is conn.} \\ \& \text{ by induction}}}{\leq} 1 + (\beta - 1) + 1 = 1 + \beta$$

disconnected cases are similar...

Part 2 of Artin-Winters argument

Given C/K, R, $k = \bar{k}$. Pick prime l not dividing $c(g)$ and prime to $\mathrm{char}(k)$ where $g = g(C)$.

Let K' be a finite separated extn of K s.t. $\mathrm{Pic}(C_{K'})[l] = (\mathbb{Z}/l\mathbb{Z})^{2g}$ and $C(K') \neq \phi$.

Replace R, K, C by $R', K', C_{K'}$, we will show C has semi-stable reduction.

Pick

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(K) \arrow[r, hook] & \mathrm{Spec}(R) \end{tikzcd}$$

regular, flat, proper model w/o (-1) curve

we will show this model is semi-stable.

Step 1. existence of rat'l point implies

- $\gcd(r_i) = 1$ • $h^0(X_s, \mathcal{O}_{X_s}) = 1$.

w/ flatness $\Longrightarrow h^1(X_s, \mathcal{O}_{X_s}) = g$

add a component a time ⓘ? $\longrightarrow h^1(X_s, \mathcal{O}_{X_s}) \geq h^1((X_s)_{red}, \mathcal{O})$ w/ equality iff X_s reduced.

By our choice of l, we have

$$2g - \beta \leq \dim_{\mathbb{F}_l} \mathrm{Pic}(X_s)[l] \overset{\text{easy}}{=\!=} \dim_{\mathbb{F}_l} \mathrm{Pic}((X_s)_{red})[l].$$

Step 2: Set $Y = (X_s)_{red}$, then we get

(⊛)

$$0 \to \Gamma(Y, \mathcal{O}_Y^*) \to \prod_{i=1}^{n} \Gamma(C_i, \mathcal{O}_{C_i}^*) \to \prod_{\substack{x \in C_i \cap C_j \\ i \neq j}} \mathcal{O}^*_{C_i \cap C_j, x}$$

$$\hookrightarrow \mathrm{Pic}^{0,0}(Y) \to \prod_i \mathrm{Pic}^0(C_i) \to 0 .$$

Facts: $\dim_{\mathbb{F}_\ell} \mathrm{Pic}^0(C_i)[\ell] \le g(C_i) + p_a(C_i)$ w/ equality iff C_i nodal (Oort).

$\dim_{\mathbb{F}_\ell} (\mathcal{O}^*_{C_i \cap C_j, x})[\ell] = 1$ and equal to $\dim_k(\mathcal{O}_{C_i \cap C_j, x})$ iff $C_i \cap C_j$ at x transversely.

Hence $(2g - \beta) \le \dim_{\mathbb{F}_\ell}(\mathrm{Pic}(Y)[\ell]) \le \sum(g(C_i) + p_a(C_i)) + \beta$.

So $2g \le \sum(g(C_i) + p_a(C_i)) + 2\beta$.

do ⊛ w/ $\mathcal{O} = \mathbb{G}_a$, $\dim_k(H^1(Y, \mathcal{O}_Y)) = \sum p_a(C_i) + \beta$

$g = \dim_k H^1(X_s, \mathcal{O}) \ge \dim_k H^1(Y, \mathcal{O}_Y) = \sum p_a(C_i) + \beta$

$\Longrightarrow \quad 2\left(\sum p_a(C_i) + \beta\right) \le \sum(g(C_i) + p_a(C_i)) + 2\beta$

So $\sum p_a(C_i) \le \sum g(C_i)$

Hence every ineq. above is equality.

Néron-Ogg-Shafarevich.

R Dedekind domain (dvr) w/ f.f. K. A is an abel. vty/K.

Defn A Néron model for A is a smooth gp scheme /R $G \longrightarrow \mathrm{Spec}(R) = S$, w/ $G_K \cong A$ and s.t. the following holds:

(∗) If X smooth /$\mathrm{Spec}(R)$, then any rat'l map

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr] & & G \arrow[dl] \\ & S & \end{tikzcd}$$

extends to a morphism.

or (alternative weaker thing)

(†) If X is smooth /$\mathrm{Spec}(R)$, then any $X_K \to A$ extends to $X \to G$.

Rmk In both cases we have, if R is a dvr, then

$$A(K^{sep})^I = A(\mathrm{f.f.}(R^{s.h.})) \underset{\substack{\uparrow \\ \text{get from } (*) \text{ or } (\dagger).}}{=} G(R^{s.h.}) \xrightarrow{sp} G(k^{sep})$$

$(K^{sep})^I = \mathrm{f.f.}(R^{s.h.})$.

Thm (Weil) Finite type Néron model for abelian varieties exist over Dedekind domains.

Thm (Raynaud) loc. of finite type smooth models with (†) exist for <u>semi-abelian vty</u>. They are also called Néron models.

Example. $\mathbb{G}_{m,K}$ w/ R dvr. Pick $\pi \in R$ uniformiser.

Set "$G = \bigcup_{n \in \mathbb{Z}} \pi^n \mathbb{G}_{m,R}$". Glue along generic fibers.

Then we have $K^\times = \mathbb{G}_{m,K}(K) = G(R) = \bigcup_{n \in \mathbb{Z}} \pi^n \cdot R^\times$.

book of Borel on linear alg. gp…

Example: Elliptic curve w/ mult. reduction and $v(\Delta)=1$.
Assume we have a Weierstrass equation $/R$ w/ discriminant, $\Delta=\pi\cdot u$.
Then
$$\begin{array}{c} G \\ \downarrow \\ \mathrm{Spec}(R) \end{array} = \left\{\begin{array}{l}\text{closed subscheme}\\ \text{of } \mathbb{P}^2_R \text{ defined by}\\ \text{Weierstrass equation}\end{array}\right\} \setminus \left\{\begin{array}{c}\text{singular point}\\ \text{of special}\\ \text{fiber}\end{array}\right\}.$$

<u>Group schemes / fields</u>

Chevalley's Thm If G/K smooth connected, then $\exists$ $1\to L\to G\to A\to 1$ where L is connected linear alg. gp., A abelian vty.

Rmk If K perfect, then L is smooth.

Thm. If L/K commutative smooth connected linear alg. gp, then
$0\to U\to L\to T\to 0$ w/ U <u>unipotent</u> and T <u>torus</u>.

Rmk If K perfect, then $L=U\times T$ canonically.

Defn. Unipotent $\Longleftrightarrow$ sits in $\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1\end{pmatrix} \leq GL_n$.

Torus T: $T\underset{K}{\otimes}\bar{K} \cong \mathbb{G}_{m,\bar{K}}^{\oplus r}$ for some r.

A gp scheme G/K is <u>semi-abelian</u> iff G is abelian and an extn of an abelian variety by a torus. A gp scheme G/S is (semi)-abelian if $G\to S$ is smooth and all fibers are (semi)-abelian.

Defn. R dvr w/ f.f. K, A/K abel. vty. We say A has <u>good reduction</u> iff its Néron model is an abelian scheme

$\boxed{\Rightarrow ?}$ $\Longleftrightarrow$ $\exists$ abelian scheme $G\longrightarrow \mathrm{Spec}(R)$ w/ $A\cong G_K$.

Thm. $A/K \supseteq R$ dvr, A has good reduction iff $\exists\, \ell$ prime to $\mathrm{char}(\kappa)$,
(Néron-Ogg-Shafarevich) s.t. $I \subset A[K^{sep}][\ell^\infty]$ is trivial.

"proof": $\Leftarrow$: $A(K^{sep})^I = A(\underset{K^{s.h.}}{\underset{\|}{f.f.\,R^{sh.}}}) = G(R^{s.h.}) \xrightarrow[\substack{\uparrow\\ \text{not inj.}}]{sp} G(\kappa^{sep})$.

$\boxed{\hookrightarrow ?}$ $(\mathbb{Z}/\ell^n\mathbb{Z})^{2g} \underset{\substack{\uparrow \\ \text{I acts trivially} \\ \text{by assumption.}}}{\cong} A(K^{sh.})[\ell^n] \xhookrightarrow{sp} G_s(\kappa^{sep})[\ell^n]$.

$G\longrightarrow \mathrm{Spec}(R)$ f.t. $\Longrightarrow [G_s : G_s^\circ] < \infty$.
$\Longrightarrow \# G_s^\circ(\kappa^{sep})[\ell^n] \geq \ell^{2gn - \text{constant}}$

Structure of gp sch$/\bar{\kappa}$:
$0\longrightarrow L\longrightarrow G_s\otimes\bar{\kappa}\longrightarrow \overset{\text{abel. vty.}}{B}\longrightarrow 0$
$0\longrightarrow U\longrightarrow L\longrightarrow T\longrightarrow 0.$
$g = \dim U + \dim T + \dim B = u+t+b.$
$\# B(\bar{\kappa})[\ell^n] = \ell^{2bn}$
$\# T(\bar{\kappa})[\ell^n] = \ell^{tn}$
$\# U(\bar{\kappa})[\ell^n] = 1$

Combine everything
$u=t=0.$

Much harder thm: I acts unipotently on $T_\ell(A)$ iff A has semi-abelian reduction which means conn. comp. of special fiber is semi-abelian.
(in SGA 7 they call this stable reduction).

abelian vty $A/K \supseteq R$ dvr. $R/(\pi) = \kappa$.

Main Thm A has semi-abelian reduction iff $\exists\, \ell \neq \mathrm{char}(\kappa)$ s.t. I acts unipotently on $T_\ell A$.

Rmk. D-M proved $A = \mathrm{Jac}(C)$ has semi-~~stable~~ abelian reduction $\Leftrightarrow$ C has semi-~~abelian~~ stable reduction. Combining these, DM proved semi-stable reduction for curves.

proof of main thm "$\Leftarrow$": assume R complete w/ finite κ. $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

$$0 \to I \to \mathrm{Gal}(K^{sep}/K) \to \underset{\overset{\|}{\mathrm{Gal}_\kappa}}{\hat{\mathbb{Z}}} \to 0$$

$\sigma \in I$ top. generator of $I_t = I/P \cong \prod_{\ell' \neq \mathrm{char}\,\kappa} \mathbb{Z}_{\ell'}$.

$\tau \in \mathrm{Gal}(K^{sep}/K)$ maps to arith-Frob in $\mathrm{Gal}_\kappa = \hat{\mathbb{Z}} \ni 1$.

$$\boxed{\tau\sigma\tau^{-1} = \sigma^q \mod P}$$

Let r = greatest integer s.t. $(\sigma-1)^r \neq 0$ on $V_\ell(A)$

$$\begin{tikzcd}
(V_\ell)_I \arrow[d, two heads] & (V_\ell)^I \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] \\
\mathrm{Gal}_\kappa \circlearrowright Q = V_\ell/\ker((\sigma-1)^r) \arrow[r, "\simeq", "(\sigma-1)^r"'] & \mathrm{Im}((\sigma-1)^r) =: S \circlearrowleft \mathrm{Gal}_\kappa
\end{tikzcd}$$

$Q(r) \xrightarrow{\sim} S$ as Gal_κ reps.

$$\tau(\sigma-1)^r = (\tau\sigma\tau^{-1}-1)^r\tau = (\sigma^q-1)^r\tau = q^r(\sigma-1)^r\tau.$$

Lemma The eigenvalue of τ^{-1} on $(V_\ell)^I$ are either q-Weil #s of weight -2 or -1.

pf. $(V_\ell)^I = V_\ell(G_s^\circ)$, coming from torus or abelian vty part, gives weight -2 or -1.

Lemma. The eigenvalues of τ^{-1} on $(V_\ell)_I$ are either q-Weil #s of weight 0 or -1.

pf. $(V_\ell(A))_I \cong ((V_\ell(A^\vee))^I)^\vee(1)$ by Weil pairing.
So we are done.

Conclusion $0 \leq r \leq 1$.

$r=0 \Rightarrow$ Néron-Ogg-Shafarevich says good reduction

$r=1 \Rightarrow$ Jordan blocks are (1) or $\begin{pmatrix}1&1\\0&1\end{pmatrix}$

$\Rightarrow 2g = \#(1) + 2\#\begin{pmatrix}1&1\\0&1\end{pmatrix}$.

Loking at weights, we see dimn of torus part $\geq \#\begin{pmatrix}1&1\\0&1\end{pmatrix}$

Use $(V_\ell)^I = V_\ell(G_s^\circ)$, get $\underset{\overset{\|}{\dim(\ker(\sigma-1))}}{2b+t} = 2g - \underset{\overset{\|}{\dim \mathrm{Im}(\sigma-1)}}{\#\begin{pmatrix}1&1\\0&1\end{pmatrix}} \geq 2g-t$.

$g = u+t+b$.

where $u = \dim U$, $t = \dim T$, $b = \dim B$, and

$$0 \to L \to G_s^\circ \to B \to 0$$
$$0 \to U \to L \to T \to 0.$$

So we are done.